MATHIGH

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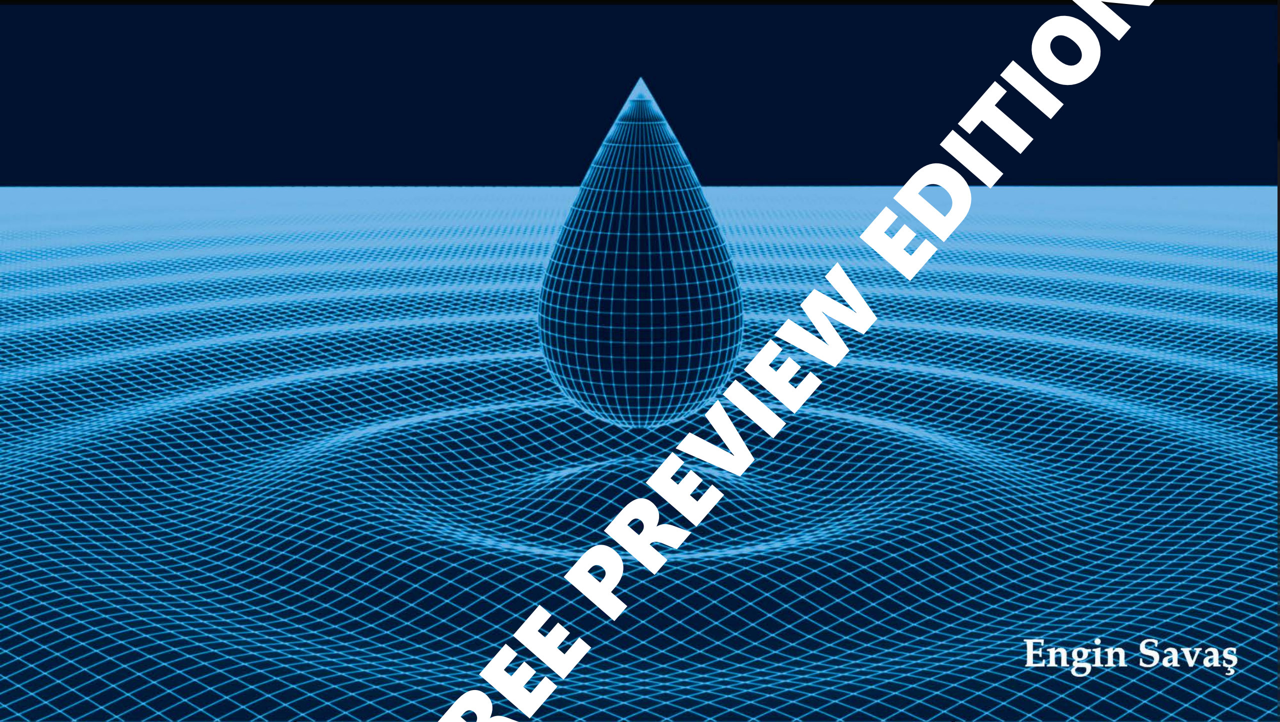
AB CONTENT & BC CONTENT

AP CALCULUS
The Ripple Effect:

A Revolutionary Approach to AP Calculus AB/BC



Engin Savaş

435 Pages of Expert Instruction

Calculator Mastery Hub Section

FRQ Strategy Room Included

Step-by-Step Concept Paths &Checkpoints

MATHIGH

AP CALCULUS

The Ripple Effect:

A Revolutionary Approach to AP Calculus AB/BC

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A Note from the Author

"Books are silent teachers; they pack patience, wisdom, and the universe into our pockets."

With over 20 years of teaching experience and 4 years as a mathematics editor, I set out to create the most student-friendly AP Calculus book possible.

This book brings together insights from nearly 10 books I've written and countless observations from students I've guided through the AP Calculus journey.

More than just a textbook, this is designed to be your interactive learning companion—a mentor that walks with you through the challenges and milestones of your preparation.

I sincerely hope it becomes not just a resource, but your most reliable teacher throughout this process.

–Engin Savaş, Author & Math Educator

Acknowledgements

This book is the result of countless hours of thought, revision, and dedication to helping students truly master AP Calculus—not just for the exam, but for life as mathematical thinkers.

I would like to thank all the students who challenged me with their curiosity, persistence, and insightful questions. Their determination has been a source of inspiration throughout this process. I am also grateful to my colleagues and fellow educators who constantly raise the bar for excellence in teaching. Their critical feedback and encouragement pushed me to refine every detail. Special thanks to my family and loved ones for their unwavering support, patience, and belief in this project from the very beginning. Lastly, I deeply appreciate the team at Desmos Studio for creating tools that empower mathematical learning and make visual intuition accessible to everyone. Some visuals in this book were created using the Desmos Graphing Calculator, used with permission of Desmos Studio PBC.

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Category A– Graphical Analysis Skills

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Calculus at the Core of Our World

Calculus is far more than a subject in school—it is the language of change. From the falling of an apple to the orbit of the planets, it explains how the universe evolves.

Sience: Newton created calculus to describe motion. Without derivatives and integrals, predicting planetary orbits would be impossible.

Medicine: Doctors analyze the rate of blood flow or heartbeats through derivative-based models.

Economics: Every marginal cost, marginal revenue, and optimization problem is rooted in calculus.

At its heart, calculus gives us the ability to move from the instantaneous to the infinite. By mastering these foundations, you gain access to the same mathematical lens that scientists, doctors, and economists rely on every day.

Core Units

Master the Foundations, Build Exam-Ready Skills

Why This Section Matters

The Core Units are the backbone of your AP Calculus preparation. Here, you'll master every key concept—limits, derivatives, integrals, and beyond—through a logical, step-by-step progression designed to build both accuracy and speed.

While advanced strategies are powerful, they only work if your foundation is rock solid. This section ensures that when you encounter any problem on the exam, you'll have the knowledge, techniques, and confidence to solve it.

What You'll Learn

Each Core Unit blends clear explanations, visual aids, and targeted practice. You'll gain:

A deep understanding of every major AP Calculus topic

Proven methods to solve both Multiple-Choice and Free-Response questions

Pattern recognition skills to identify problem types quickly

Step-by-step strategies that work under time pressure

How to Approach This Section

Study the left page for concept explanations and model examples.

Apply on the right page with carefully selected practice problems.

Complete the Checkpoint tests at the end of every few units to track your progress.

Review mistakes immediately and reattempt similar problems to reinforce mastery.

A Strategic Reminder

The AP Calculus exam rewards precision, reasoning, and adaptability. Core Units are your training ground for all three—build them here, and every advanced section in this book will work twice as hard for you.

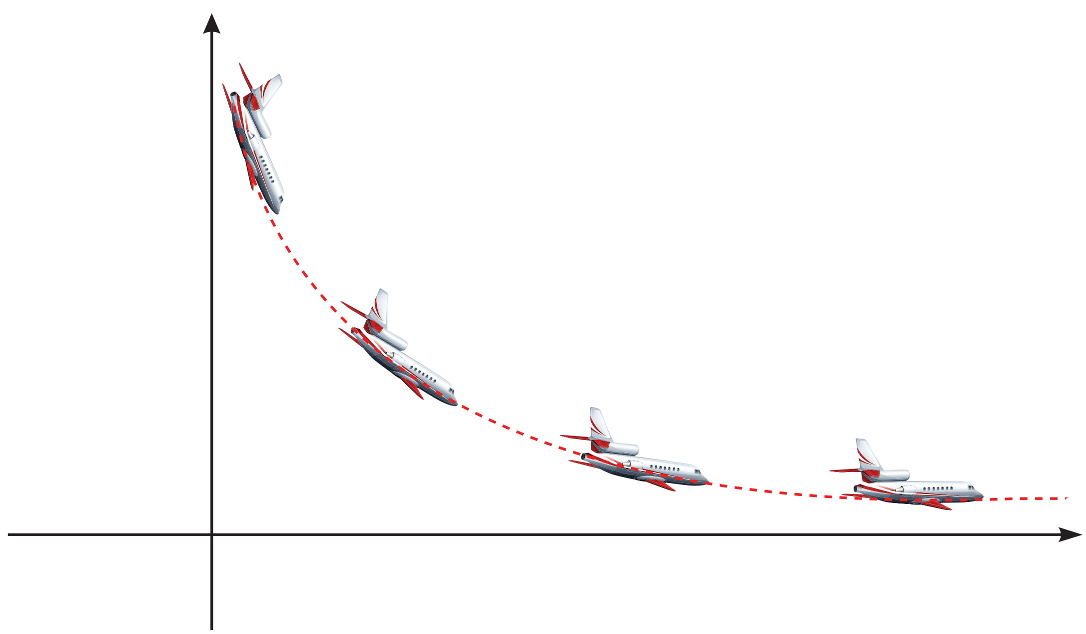
UNIT 1: LIMIT AND CONTINUITY

"The essence of mathematics is not to make simple things complicated, but to make complicated things simple." - S. Gudder

The Big Picture: Welcome to the foundation of all calculus. This unit introduces the concept of the limit, which is the language we use to describe how functions behave as they get infinitely close to a point. By understanding limits, we can begin to analyze functions with holes, jumps, and infinite breaks, setting the stage for the powerful ideas of the derivative and the integral.

Key Missions:

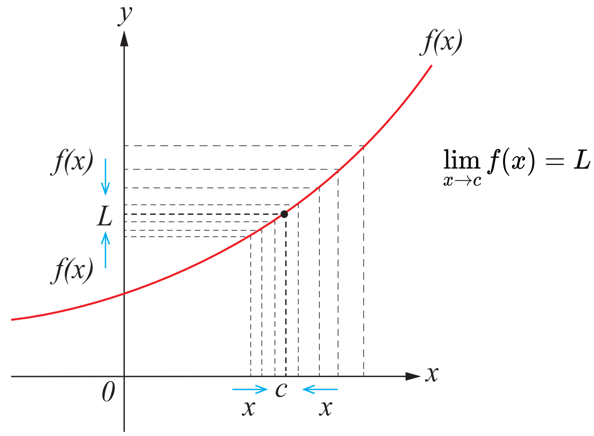
- Mastering the three ways to find a limit: graphically, numerically, and analytically.
- Understanding the formal definition of continuity.
- Using the Intermediate Value Theorem to prove the existence of solutions.



The Core Principle: The Simplest Path to a Limit

For most "well-behaved" and continuous functions (like polynomials), the value a function *approaches* as x gets close to a point c is simply the function's *actual value* at c. Therefore, the first, fastest, and most important method you must always try is *Direct Substitution*. If plugging in the value gives you a real, defined number, you have found the limit.

The Core Graphic: A Smooth Approach



- **Visual**: A smooth, continuous curve passing through a labeled point (c, L). Arrows along the curve should approach this point from both the left and the right.
- **Purpose**: The visual reinforces that for continuous functions, the limit (the approach) is identical to the function's value (the point).

The Toolkit: When to Use & When to Stop

Use Direct Substitution If...	STOP! Find Another Method If...
The function is a polynomial, rational, exponential, trigonometric, or root function andthe point c is in the domain.	Substitution results in 0/0 or ∞/∞ (These indeterminate forms will be covered in Topics 1.9 & 1.10).
You are performing a quick check before trying more complex methods.	The denominator becomes zero (but the numerator does not), indicating a vertical asymptote (This will be covered in Topic 1.4).

Pitfalls & Pro-Tips

- **Pro-Tip:** Always check the denominator first. A zero in the denominator immediately tells you that direct substitution will not work.
- **Pitfall:** Applying complex methods like L'Hôpital's Rule before trying direct substitution. This is a common waste of time on exams.
- **Pitfall:** Forgetting domain restrictions, such as plugging $x = 0$ into $\ln(x)$. Always ensure the point is in the function's domain.

Worked Example: Direct Substitution in Action

Problem: Find the limit: $\lim_{x \to 2} \left(x^3 - 2x^2 + 4x - 1\right)$

Solution:

1. **Check the function type:** The function $f(x) = x^3 - 2x^2 + 4x - 1$ is a polynomial. Polynomials are continuous everywhere. Therefore, direct substitution is the correct method.
2. **Substitute the value:** Replace every *x* in the function with $(2)^3 - 2(2)^2 + 4(2) - 1$
3. **Calculate the result:** $8 - 2(4) + 8 - 1 = 7$

Answer: The limit is **7**.

Practice Exercises

Instructions: Find the value of each limit using direct substitution.

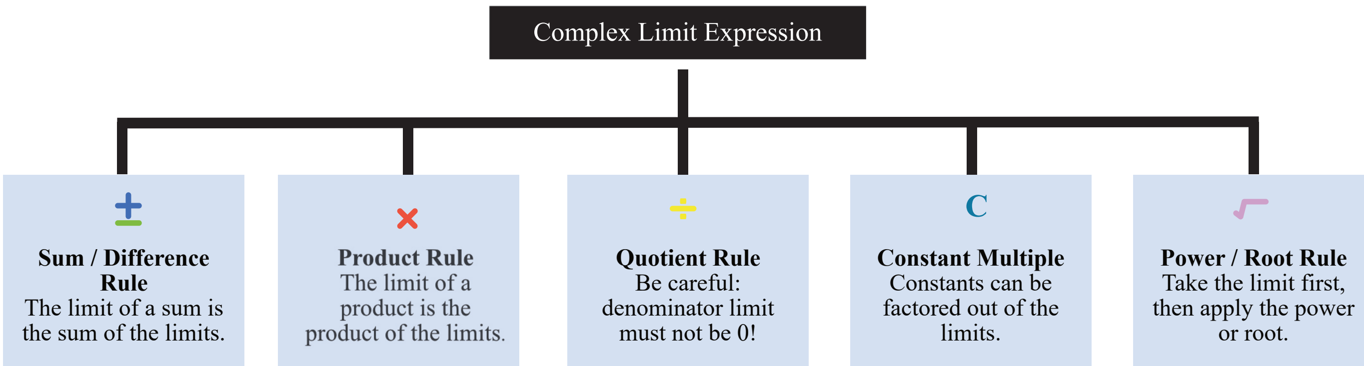
1. $\lim_{x \to -1} \frac{3x+5}{x-2}$	**3.** $\lim_{x \to \pi} [\sin(x) + \cos(x)]$
2. $\lim_{x \to 0} \left(e^{2x+1}\right)$	**4.** $\lim_{x \to 2} \sqrt{2x+1}$

The Core Principle: The Algebra of Limits

We don't need to evaluate every complex limit from scratch. If a complex function is built from simpler functions using basic arithmetic, we can find its limit by performing the same arithmetic on the limits of the simpler pieces.

In short: **The limit of a sum is the sum of the limits**. This principle applies to most arithmetic operations.

The Core Graphic: The Limit Laws



- **Visual**: A central box labeled "Complex Limit" with arrows pointing to smaller boxes labeled "Sum Rule," "Product Rule," "Quotient Rule," etc.
- **Purpose**: The visual shows that a complex limit can be broken down into simpler parts, each handled by a specific "tool" or "law.

The Complete Toolkit: The Limit Laws

Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.					
Property	Sum / Difference Rule	Product Rule	Quotient Rule	Constant Multiple Rule	Power / Root Rule
The Rule	$\lim_{x\to a}[f(x)+g(x)] = L+M$	$\lim_{x\to a}[f(x)\cdot g(x)] = L\cdot M$	$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}$ (provided $M \neq 0$)	$\lim_{x\to a}[cf(x)] = c\cdot L$	$\lim_{x\to a}[f(x)]^n = L^n$

Pitfalls & Pro-Tips

- **Critical Pitfall:** The Quotient Rule only works if the limit of the denominator (M) is not zero. Applying the rule when $M = 0$ is a major conceptual error.
- **Pro-Tip:** Before applying any rule, it's good practice to consider the limits of the individual pieces first. If any sub-limit does not exist or is infinite, the rules may not apply directly.
- **Pro-Tip:** For root rules with an even index (like square roots), the limit L must be non-negative.

Worked Example: Applying the Limit Laws

Problem: Given that $\lim_{x\to 1} f(x) = 8$ and $\lim_{x\to 1} g(x) = -2$, find $\lim_{x\to 1}[2f(x) - 3g(x)]$.

Solution:

1. Apply the Difference Rule: Separate the limit into two parts $\lim_{x\to 1}[2f(x)] - \lim_{x\to 1}[3g(x)]$.
2. Apply the Constant Multiple Rule to both terms. $2\lim_{x\to 1} f(x) - 3\lim_{x\to 1} g(x)$.
3. Substitute the known limit values. 2(8) – 3(–2)
4. Calculate the result: 16 + 6 = 22

Answer: The limit is **22**.

Practice Exercises

Instructions: Use the given information below to find the value of each limit.

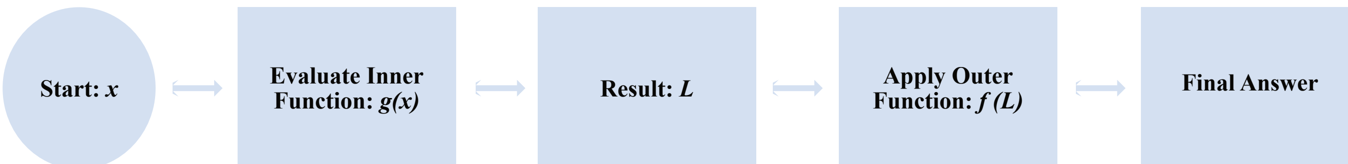
Given: $\lim_{x\to 3} f(x) = 6$, $\lim_{x\to 3} g(x) = -3$, $\lim_{x\to 3} h(x) = 4$.

1. $\lim_{x\to 3}[g(x) - 2f(x)]$	**3.** $\lim_{x\to 3} \dfrac{f(x)\cdot g(x)}{h(x)}$
2. $\lim_{x\to 3}[xf(x)]$	**4.** $\lim_{x\to 3} \sqrt{h(x) + g(x) + 6}$

The Core Principle: The "Inside-Out" Rule

When we have a function "nested" inside another, like $f(g(x))$, we can find its limit by working from the **inside out**. If the outer function f is continuous, we can "pass the limit inside" to the inner function g first. This means you find the limit of the inner function, get the result, and then apply the outer function to that result.

The Core Graphic: A Flowchart for Limits



- **Visual**: A flowchart showing the process:
  1. Start: $x \to c$
  2. Process inner function: $g(x)$
  3. Get result: L
  4. Process outer function: $f(L)$
  5. Final Answer.
- **Purpose**: This chart visually breaks down the process, showing how the limit evaluation flows from the inner function to the outer one.

The Complete Toolkit: The Formal Rule

For a composite function $f(g(x))$, if the outer function f is continuous at the value that $g(x)$ approaches, then you can bring the limit inside: $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$

The Method:

1. Solve the Inner Limit: First, calculate the limit of the inside function, $g(x)$. Let's call the result L.

2. Apply the Outer Function: Plug the result L into the outer function, $f(x)$. The final answer is $f(L)$.

Pitfalls & Pro-Tips

• **Critical Pitfall:** Do not apply the outer function before you find the limit of the inner function.

You must evaluate the inner limit to a single number first.

• **Pro-Tip:** You are implicitly using this rule all the time! When you solve limits of most standard functions, you are using the inside-out rule without realizing it.

• **Pitfall:** Confusing a composite function limit, $f(g(x))$, with a product of functions, $f(x) \cdot g(x)$.

Worked Example: The Inside-Out Method in Action

Problem: Find the limit: $\lim_{x \to 4} \sqrt{x^2 + 9}$

Solution:

1. **Identify Inner and Outer Functions:**
   - Inner function: $g(x) = x^2 + 9$
   - Outer function: $f(u) = \sqrt{u}$
2. **Find the Inner Limit:** First, evaluate the limit of the inner function as $x \to 4$.
   $\lim_{x \to 4} \left(x^2 + 9\right) = (4)^2 + 9 = 16 + 9 = 25$
3. **Apply the Outer Function to the Result:** Now, apply the outer function *f(u)* to the result from Step 2, which is 25.$f(25) = \sqrt{25} = 5$

Answer: The limit is **5.**

Practice Exercises

Instructions: Find the value of each limit using the rule for composite functions.

1. $\lim_{x \to \pi} \cos(\sin(x))$	**3.** $\lim_{x \to e} \ln\left(x^3\right)$
2. $\lim_{x \to 1} \left(x^3 + 2x + 1\right)$	**4.** $\lim_{x \to \frac{1}{\sqrt{2}}} \arctan\left(\frac{2x^2 - 1}{2x^2 + 1}\right)$

CHECKPOINT 1.1 – Substitution & Limit Properties

Instructions: Answer the following questions to check your understanding of the first three topics: direct substitution, limit properties, and composite limits.

1. What is the value of the limit $\lim_{x\to -1}\left(x^3 - 2x^2 + 5\right) = ?$

(A) 2 (B) 4

(C) 8 (D) Does Not Exist

2. Given that $\lim_{x\to 5} f(x) = 10$ and $\lim_{x\to 5} g(x) = -2$

what is $\lim_{x\to 5} \frac{g(x)}{f(x)} = \ ?$

(A) –5 (B) $-\frac{1}{5}$

(C) 8 (D) Does Not Exist

3. What is the value of the limit $\lim_{x\to 0}(\cos(x) + 1)^3 = ?$

(A) 0 (B) 1

(C) 8 (D) 27

4. What is the very first method you should always attempt when evaluating any limit?

(A) Factoring

(B) Direct Substitution

(C) Using a graph

(D) The Squeeze Theorem

Answers & Feedback Loop

Answers and Explanations

1. (A) 2

The function is a polynomial, so we can use **Direct Substitution**.

$(-1)^3 - 2(-1)^2 + 5 = -1 - 2(1) + 5$

$= -1 - 2 + 5 = 2$

2. (C) $-\frac{1}{5}$

By the **Quotient Rule** for limits, we can divide the limits of the individual functions, as long as the denominator's limit is not zero.

$$\frac{\lim_{x \to 5} g(x)}{\lim_{x \to 5} f(x)} = \frac{-2}{10} = -\frac{1}{5}$$

3. (B) 8

This is a **Composite Limit**. First, find the inner limit: $\lim_{x \to 0}(\cos(x) + 1) = \cos(0) + 1 = 1 + 1 = 2$

Then, apply the outer function (cubing) to the result $(2)^3 = 8$

4. (B) Direct Substitution.

This is the fastest and most fundamental method. It should always be the first step, as it works for any function that is continuous at the limit point.

Self-Assessment Guide

All Correct: Excellent! You have a solid grasp of these foundational concepts. You are ready for the next block of topics.

One Mistake: Great work. Briefly review the topic page for the question you missed, and then you can proceed.

More Than One Mistake: Stop and Review. It is crucial to master these foundational concepts before moving on. Please carefully re-study the relevant topic pages (1.1 - 1.3) before proceeding.

The Core Principle: Left vs. Right & Vertical Asymptotes

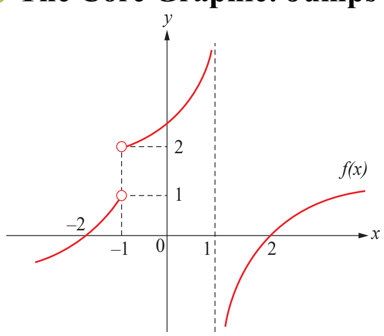
This topic covers two critical ideas for understanding limits where direct substitution fails:

1. **One-Sided Limits:** For a general, two-sided limit $\lim_{x \to c} f(x)$ to exist, the function must approach the **same *y*-value** from both the left and the right.

- **Left-Hand Limit:** $\lim_{x \to c^-} f(x)$ (approaching c from numbers smaller than c).
- **Right-Hand Limit:** $\lim_{x \to c^+} f(x)$ (approaching c from numbers larger than c).

2. **Infinite Limits:** When direct substitution results in a **(non-zero number) / 0** form, it signals a **vertical asymptote**. The limit is not a finite number; it will be +∞ or -∞.

The Core Graphic: Jumps and Asymptotes



- **Visual**: A single graph showing two key behaviors. At $x = -1$, there is a jump, where the function approaches a y-value of 1 from the left and a y-value of 2 from the right. At $x = 1$, there is a vertical asymptote, where the function approaches positive infinity (+∞) from the left and comes up from negative infinity (−∞) on the right.

- **Purpose**: This single visual powerfully demonstrates why we need one-sided limits to describe behavior at a jump (like at $x = -1$) and how to analyze the different infinite behaviors on either side of a vertical asymptote (like at $x = 1$).

The Toolkit: Analyzing Signs for Infinite Limits

When you see a $c/0$ form, the key is to determine if the denominator is approaching zero from the positive side (0^+) or the negative side (0^-).

Numerator Sign	Denominator Approach	Result
Positive ($c > 0$)	Approaches from positive (0^+)	$+\infty$
Positive ($c > 0$)	Approaches from negative (0^-)	$-\infty$
Negative ($c < 0$)	Approaches from positive (0^+)	$-\infty$
Negative ($c < 0$)	Approaches from negative (0^-)	$+\infty$

Pitfalls & Pro-Tips

- **Critical Pitfall:** Do not automatically write "DNE" (Does Not Exist) for a $c/0$ limit. While the limit doesn't exist as a *finite number*, a more precise answer like +∞ or −∞ is often required. If both sides approach +∞, the limit is +∞.
- **Pro-Tip:** The most important step is analyzing the sign. Think: "If x is slightly bigger than c, is the denominator a small positive or small negative number?"
- **Pitfall:** Confusing a $c/0$ case (infinite limit) with a 0/0 case (indeterminate form). A 0/0 limit might be a finite number after algebraic simplification; a $c/0$ limit will always be infinite or DNE.

Worked Example: Analyzing an Infinite Limit

Problem: Find the limit: $\lim_{x \to 3} \frac{1}{(x-3)^2}$

Solution:

1. Check for Asymptote: Direct substitution gives : $\frac{1}{(3-3)^2}$ which is 1/0. This confirms a vertical asymptote at $x = 3$.

2. Analyze the Right-Hand Limit ($x \to 3^+$): Pick a number slightly greater than 3 (e.g., 3.1). The denominator is $(3.1-3)^2 = (0.1)^2$, which is a small **positive** number (0^+). The result is 1 / (small positive), which approaches $+\infty$.

3. Analyze the Left-Hand Limit ($x \to 3^-$): Pick a number slightly less than 3 (e.g., 2.9). The denominator is $(2.9-3)^2 = (-0.1)^2$, which is also a small **positive** number (0^+). The result is 1 / (small positive), which also approaches $+\infty$.

4. **Conclusion:** Since both the left-hand and right-hand limits approach $+\infty$, the overall limit is $+\infty$.

Answer: The limit is ∞.

Practice Exercises

Instructions: Find the value of each limit. If a two-sided limit does not exist because the sides differ, write "DNE".

1. Find $\lim_{x \to 2^-} \frac{5}{x-2}$	**3.** Find $\lim_{x \to 1} \frac{-2}{(x-1)^4}$
2. Find $\lim_{x \to 4} \frac{5}{x-4}$	**4.** Find $\lim_{x \to 0^+} \ln(x)$

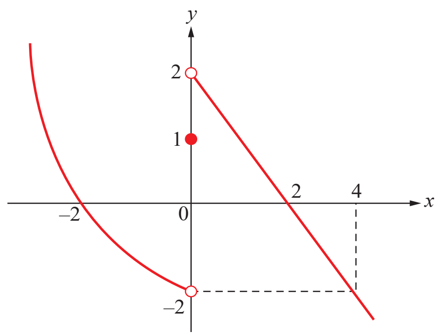
The Core Principle: Check Your Location

Finding the limit of a piecewise function depends entirely on **where** you are evaluating the limit relative to the function's **"break points"** (the x-values where the rule changes).

1. Inside an Interval: If your limit point c is **not** a break point, the process is easy. Simply usebthe one piece of the function that applies to that interval and evaluate the limit.

2. At a Break Point: If your limit point c **is** a break point, you **must** evaluate the left-hand and the righthand limits separately. The overall limit exists if and only if these two one-sided limits are equal.

The Core Graphic: The "Break Point"



• **Visual**: Draw a function that follows a parabola up to $x = 0$, ending in an open circle at $y = –2$. At $x = 0$, there is a filled circle at $y = 1$. For $x > 0$, the function follows a line starting with an open circle at $y = 2$.

• **Purpose**: This visual makes it obvious why approaching the break point $x = 0$ from the left (using the parabolic rule) and from the right (using the linear rule) are two separate calculations that may yield different y-values. It also highlights that the value of the limit at a breakpoint may be different from the actual function value at that point.

The Complete Toolkit: The Formal Process at a Break Point

To find the limit of a piecewise function $f(x)$ at a break point c, follow these steps:

1. **Find the Left-Hand Limit (LHL):** Use the piece of the function defined for $x < c$.
   Calculate $\lim_{x \to c^-} f(x)$
2. **Find the Right-Hand Limit (RHL):** Use the piece of the function defined for $x > c$.
   Calculate $\lim_{x \to c^+} f(x)$
3. **Compare:**
   • If LHL = RHL, the limit exists and is equal to this value.
   • If LHL ≠ RHL, the limit Does Not Exist (DNE).

Pitfalls & Pro-Tips

• **Critical Pitfall:** Confusing the limit at a point, $\lim_{x \to c} f(x)$ with the function's actual value at that point, f(c). The limit is about the **approach**, not the location of the solid dot.

• **Pro-Tip:** For "find the constant k that makes the limit exist" problems, the strategy is always to set the Left-Hand Limit equal to the Right-Hand Limit at the break point and solve for k.

• **Pitfall:** Using the wrong piece of the function for a one-sided limit. Always check the inequality sign ($<$ or $>$).

Worked Example: Analyzing a Break Point

Problem: Find $\lim_{x\to 1} f(x)$ for the function: $f(x) = \begin{cases} x+3, x<1 \\ -x+5, x\geq 1 \end{cases}$

Solution: Since $x = 1$ is a break point, we must check both one-sided limits.

1. **Find the Left-Hand Limit ($x\to 1^-$):** We use the rule for $x < 1$, which is $x + 3$.
   $\lim_{x\to 1^-}(x+3) = 1+3 = 4$
2. **Find the Right-Hand Limit ($x\to 1^+$):** We use the rule for $x \geq 1$, which is $-x + 5$.
   $\lim_{x\to 1^+}(-x+5) = -1+5 = 4$
3. **Compare:** The left-hand limit (4) is equal to the right-hand limit (4).

Answer: The limit exists and is **4**.

Practice Exercises

Instructions: Find the value of each limit. If the limit does not exist, write "DNE". Let : $f(x) = \begin{cases} x^2, & x<3 \\ 2x, & x\geq 3 \end{cases}$

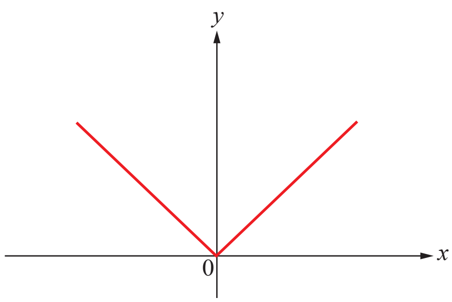
1. Find $\lim_{x\to 5} f(x)$	**3.** Find $\lim_{x\to 3} f(x)$
2. Find $\lim_{x\to 1} f(x)$	**4.** Find the value of the constant c that makes $\lim_{x\to -2} h(x)$ exist. $h(x) = \begin{cases} cx-4, & x<-2 \\ x^3+c, & x\geq -2 \end{cases}$

The Core Principle: Rewrite as a Piecewise Function

The key to solving limits involving absolute values is to eliminate the absolute value bars. We do this by rewriting the function as a **piecewise function**.

An absolute value function always "splits" at the **critical point** where the expression inside it equals zero analyzing the function's behavior to the left and right of this critical point, we can evaluate the limit.

The Core Graphic: The "V" Shape



- **Visual**: Draw the standard graph of $f(x) = |x|$. Emphasize the sharp corner (a "cusp") at the origin (0,0).
- **Labels:** Clearly label the right branch of the "V" as $\boldsymbol{y = x}$ (for $x > 0$) and the left branch as $\boldsymbol{y = -x}$ (for $x < 0$).
- **Purpose**: This visual shows that a single absolute value function is actually a piecewise function made of two different linear functions. This is the fundamental logic needed to solve these limits.

The Complete Toolkit: The Piecewise Conversion Process

1. **Find the Critical Point:** Set the expression inside the absolute value bars equal to zero and solve. This is where the function's behavior might change.
2. **Define the Pieces:** Write the piecewise definition based on the critical point. Remember the definition: $|u| = u$ if $u \geq 0$, and $|u| = -u$ if $u < 0$.
3. **Evaluate One-Sided Limits:** If the limit is at the critical point, you must find the left-hand and righthand limits separately, using the correct piece for each.

Pitfalls & Pro-Tips

- **Critical Pitfall:** Assuming the limit at a critical point must be DNE. The limit can exist if both sides approach the same value. For example, $\lim_{x \to 0} |x| = 0$ be cause both the left and right-hand limits are 0.
- **Pro-Tip: ALWAYS SUBSTITUTE FIRST!** If the limit point is *not* the critical point, the problem is usually a simple direct substitution. Don't overcomplicate it.
- **Pitfall:** Forgetting to use parentheses when negating the expression. When replacing |u| with $-u$, always write it as $-(u)$ to prevent sign errors.

Worked Example: The Two-Sided Limit in Action

Problem: Find the limit: $\lim_{x\to 2} \frac{|x-2|}{x-2}$

Solution:

1. **Rewrite as a Piecewise Function:** The critical point is $x = 2$.
   - If $x > 2$, then $x - 2$ is positive, so $|x - 2| = x - 2$.
   - If $x < 2$, then $x - 2$ is negative, so $|x - 2| = -(x - 2)$.
2. **Evaluate the Right-Hand Limit ($x\to 2^+$):** We use the $x > 2$ case. $\lim_{x\to 2^+} \frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$
3. **Evaluate the Left-Hand Limit ($x\to 2^-$):** We use the $x < 2$ case. $\lim_{x\to 2^-} \frac{-(x-2)}{x-2} = \lim_{x\to 2^+} -1 = -1$
4. **Compare the Limits:** The right-hand limit (1) is not equal to the left-hand limit (−1).

Answer: The limit **Does Not Exist**.

Practice Exercises

Instructions: Find the value of each limit. If the limit does not exist, write "DNE".

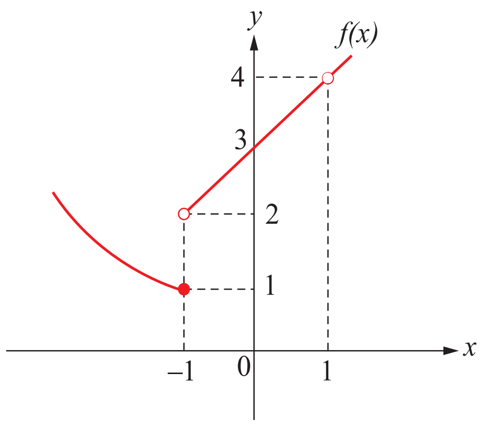
1. $\lim_{x\to 10} \lvert x-3\rvert$	**3.** $\lim_{x\to -4} \frac{x+4}{\lvert x+4\rvert}$
2. $\lim_{x\to -2} \lvert x+2\rvert$	**4.** $\lim_{x\to 3} \lvert x^2-4x+3\rvert$

The Core Principle: Intended Height vs. Actual Height

A limit from a graph answers the question: "Which y-value is the curve approaching as x gets close to a target value?" Whether the function is defined at that exact point—or takes on a different value—does not affect the limit. The limit is all about the approach.

The Golden Rule: A two-sided limit exists if and only if the curve approaches the same height from both the left and the right directions.

The Core Graphic: Holes and Jumps



- **Visual**: single graph showing two key behaviors. At $x = -1$, there is a **jump**, where the function approaches a y-value of 1 from the left and a y-value of 2 from the right. At $x = 1$, there is a **hole**, where the function approaches a y-value of 4 from both sides, but $f(1)$ is undefined.

- **Purpose**: This single visual powerfully demonstrates that a limit can exist at a **hole** (like at $x = 1$), but does not exist at a **jump** where the left and right paths do not meet (like at $x = -1$).

The Complete Toolkit: The Graphical Limit Checklist

To find $\lim_{x \to c} f(x)$ from a graph, always follow this process:

1. Find the Left-Hand Limit (LHL): Follow the graph from the left side towards $x = c$ and identify the y-value it approaches.

2. Find the Right-Hand Limit (RHL): Follow the graph from the right side towards $x = c$ and identify the y-value it approaches.

3. Compare:

   o If LHL = RHL, the limit exists and is that value.
   o If LHL ≠ RHL, the limit Does Not Exist (DNE).

Pitfalls & Pro-Tips

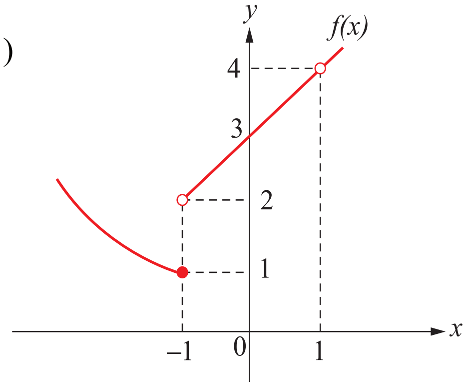
- **Critical Pitfall:** Confusing the limit, $\lim_{x \to c} f(x)$, with the function's actual value, f(c). The limit is the "intended height" the path leads to, while f(c) is the "actual height" of the solid dot.

- **Pro-Tip:** A hole can still have a limit; a jump or a vertical asymptote cannot have a finite, two-sided limit.

- **Pitfall:** Forgetting to check both sides for a two-sided limit. If they differ, the limit DNE.

Worked Example: Analyzing a Graph

Problem: Using the graph of $f(x)$, find the limit at the jump ($x = -1$) and the hole ($x = 1$).

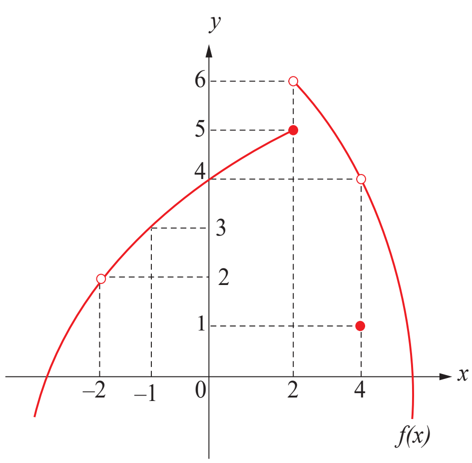


Solution:

Analysis at $x = -1$ (The Jump):	**Analysis at $x = 1$ (The Hole):**
1. **Find the Left-Hand Limit (LHL):** As $x \to -1^-$, the graph approaches a y-value of 1. 2. **Find the Right-Hand Limit (RHL):** As $x \to -1^+$, the graph approaches a y-value of 2. 3. **Compare:** Since the left-hand limit (1) does not equal the righthand limit (2), the overall limit does not exist. **Conclusion for $x = -1$:** $\lim_{x\to -1} f(x)$ Does Not Exist.	1. **LHL:** As $x \to -1^-$, the graph approaches a y-value of 4. 2. **RHL:** As $x \to -1^+$, the graph also approaches a y-value of 4 3. **Compare:** Since LHL = RHL = 4, the limit exists. (Note: The function is undefined at $f(1)$, but this does not affect the limit's existence). **Conclusion for $x = 1$:** $\lim_{x\to 1} f(x) = 4$

Practice Exercises

Instructions: Find the value of each limit. If the limit does not exist, write "DNE".



1. $\lim_{x\to -2} f(x)$

2. $f(2)$

3. $\lim_{x\to 2^-} f(x)$

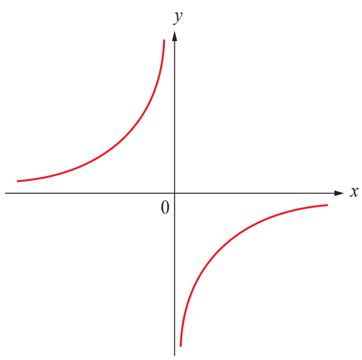
4. $f(4)$

The Core Principle: End Behavior of a Function

A limit at infinity asks: What *y-value* does the function approach as x grows infinitely large in the positive or negative direction ($x \to \infty$ or $x \to -\infty$)? This concept describes the **end behavior** of a function.

If this limit is a finite number L, then the line $y = L$ is a **Horizontal Asymptote**. It's the "destination" y-value the graph settles down to as x flies off to the left or right.

The Core Graphic: Settling Down to a Horizontal Asymptote



- **Visual**: The graph of a function like $f(x) = 1/x$ is shown, which has a horizontal asymptote at the x-axis ($y = 0$). As the graph extends to the far right ($x \to \infty$) and the far left ($x \to -\infty$), the curve gets infinitely close to this line but never touches it.

- **Purpose**: This visual powerfully demonstrates the concept of "end behavior." It shows how a function's value can approach a specific, finite number (in this case, $y = 0$) as x goes to positive or negative infinity. This "destination" y-value defines the horizontal asymptote.

The Toolkit: The Degree Comparison Rules for Rational Functions

To find $\lim\limits_{x \to \pm\infty}$ a rational function, compare the degree of the numerator ($\boldsymbol{n}$) and the degree of the denominator ($\boldsymbol{m}$).

Degree Comparison	Limit Result	Horizontal Asymptote
$n < m$ (Bottom-Heavy)	0	$y = 0$
$n = m$ (Equal Degrees)	Ratio of Leading Coefficients	$y =$ (ratio)
$n > m$ (Top-Heavy)	∞ or -∞	None

Pitfalls & Pro-Tips

- **Critical Pitfall:** Believing a function can *never* cross its horizontal asymptote. This is false. A HA only governs **end behavior**.
- **Pro-Tip:** When dealing with $x \to -\infty$ and square roots, remember that for a negative x, $\sqrt{x^2} = |x| = -x$.

  This is a common trap that can flip the sign of your answer.
- **Pitfall:** Confusing Horizontal Asymptotes ($y =$..., comes from $x \to \infty$) with Vertical Asymptotes ($x =$..., comes from $y \to \infty$).

Worked Example: The Degree Comparison Method

Problem: Find the limit: $\lim_{x \to \infty} \frac{4x^2 - 1}{3x^2 + 2x}$

Solution:

1. **Identify Form:** As $x \to \infty$, this is an ∞/∞ indeterminate form.
2. **Compare Degrees:** The degree of the numerator $(4x^2)$ is 2. The degree of the denominator $(3x^2)$ is 2.
3. **Find Ratio:** Since the degrees are equal, the limit is the ratio of the leading coefficients.

$$\frac{\text{Lead Coeff of Numerator}}{\text{Lead Coeff of Denominator}} = \frac{4}{3}$$

4. **Conclusion:** The limit exists and is a finite number. This means the function has a horizontal asymptote at y = 4/3.

Answer: The limit is **4/3**.

Practice Exercises

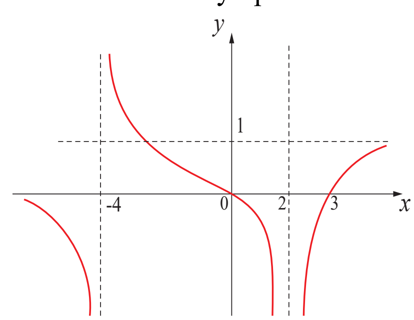
Instructions: Find the value of each limit.

1. $\lim_{x \to \infty} \frac{10x^2 - 3x + 1}{5x^2 + 2}$

2. $\lim_{x \to \infty} \frac{x^2 + 5x - 3}{x^3 + 2x}$

3. $\lim_{x \to -\infty} \frac{4x^3 + 5x}{2x^2 - 7x}$

4. For the function $f(x)$ shown in the graph below, find the equations of all horizantal asymptotes.



CHECKPOINT 1.2 – Directional & Structural Limits

Instructions: Answer the following questions to check your understanding of one-sided limits, asymptotes, piecewise functions, absolute values, and graphical analysis.

1. Given the function $f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 5x + 2, & x \geq 3 \end{cases}$

what is $\lim_{x \to 3} f(x)$?

(A) 5 (B) 8

(C) 17 (D) Does Not Exist

2. What is the value of the limit $\lim_{x \to 1} \dfrac{|x-1|}{x-1}$?

(A) –1 (B) 0

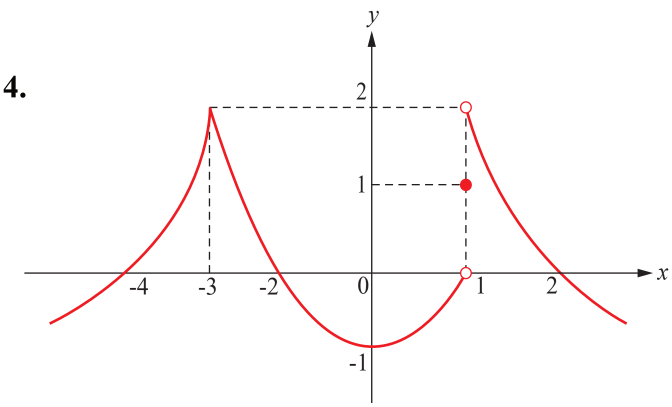
(C) 1 (D) Does Not Exist

3. What is the value of the limit $\lim_{x \to \infty} \dfrac{1 - 2x^2}{x^2 + 4}$?

(A) –2 (B) $\dfrac{1}{4}$

(C) 1 (D) ∞

4.



Based on the graph of $f(x)$ provided, what is the value of $\lim_{x \to 1^+} f(x) + \lim_{x \to -3^-} f(x)$?

(A) 2 (B) 3

(C) 4 (D) Does Not Exist

Answers & Feedback Loop

Answers and Explanations

1. (D) Does Not Exist

This is a piecewise function at a break point, so we must check both sides.

- LHL: $\lim_{x \to 3^-} (x^2 - 1) = 3^2 - 1 = 8$
- RHL: $\lim_{x \to 3^+} (5x + 2) = 5(3) + 2 = 17$

Since the left-hand limit (8) does not equal the right-hand limit (17), the limit does not exist.

2. (D) Does Not Exist

This is an absolute value limit at its critical point.

- **LHL ($x \to 1^-$):** The inside is negative, so $|x - 1| = -(x - 1)$. The limit becomes

$$\lim_{x \to 1} \frac{-(x-1)}{x-1} = \lim_{x \to 1} (-1) = -1$$

- **RHL ($x \to 1^+$):** The inside is positive, so $|x - 1| = x - 1$. The limit becomes

$$\lim_{x \to 1} \frac{(x-1)}{x-1} = \lim_{x \to 1} (1) = 1$$

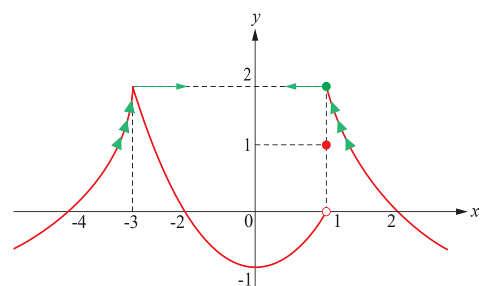
Since the one-sided limits are not equal, the limit does not exist.

3. (A) -2

This is a limit at infinity of a rational function. Since the degrees of the numerator (x^2) and Denominator (x^2) are equal, the limit is the **ratio of the leading coefficients**: $-\frac{2}{1} = -2$.

4. (C) 4

This question requires finding two separate one-sided limits from the graph and adding them.



First Limit: To find $\lim_{x \to 1^+} f(x)$, we approach $x = 1$ from the right side. The graph shows the function's y-value approaches **2**.

Second Limit: To find $\lim_{x \to -3^-} f(x)$, we approach $x = -3$ from the left side. The graph shows the function's y-value approaches **2**.

Sum: $2 + 2 = 4$.

Self-Assessment Guide

All Correct: Fantastic work! You have a strong command of the structural and directional aspects of limits. You are ready to tackle indeterminate forms.

One Mistake: Good job. Take a moment to review the topic related to your error. A quick refresher should be all you need.

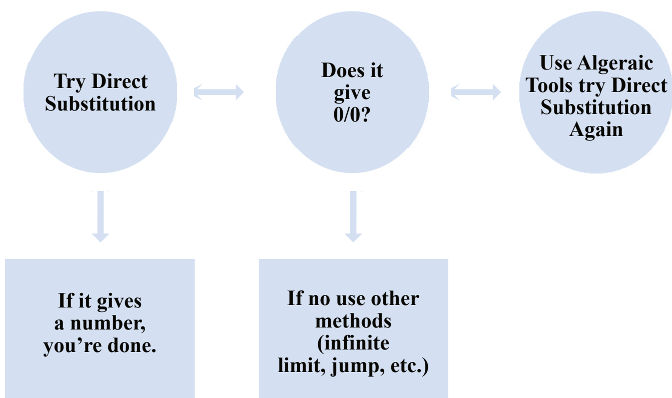
More Than One Mistake: Stop and Review. These concepts are the foundation for all of calculus. It is essential that you re-study the relevant topic pages (1.4 - 1.8) before moving forward.

The Core Principle: A Signal to Do More Work

When direct substitution into a limit problem results in the form 0/0 , this is not an answer. It is an **indeterminate form**. This result tells us nothing about the limit's value, but it signals that the function has a **hole** (a removable discontinuity) at that point.

Our goal is to use algebraic techniques to manipulate the function, cancel the term that causes the 0 in the denominator, and reveal the true limit.

The Core Graphic: The 0/0 Resolution Flowchart



- **Visual**: A flowchart showing the process:
  1. Try Direct Substitution.
  2. Result is 0/0? → YES.
  3. Apply Algebraic Manipulation (Factor, Conjugate, etc.).
  4. Simplify & Cancel.
  5. Re-evaluate the Limit.
- **Purpose**: This flowchart provides a clear, repeatable strategy for students to follow whenever they encounter the 0/0 form.

The Toolkit: Algebraic Methods for 0/0

Expression Type	Technique	Goal
Polynomials	Factoring & Canceling	Find and cancel the common $(x - a)$ term causing the issue.
A Square Root	Multiplying by the Conjugate	Use the conjugate to reveal a hidden factor that can be canceled.
Fractions within a Fraction	Simplifying Complex Fractions	Combine terms into a single fraction to allow for cancellation.

Pitfalls & Pro-Tips

- **Critical Pitfall:** Assuming that 0/0 equals 0, 1, or is "undefined". It is none of these. It is a signal to **start working**, not a final answer.
- **Pro-Tip:** If you see a polynomial, try factoring. If you see a square root, think conjugate. This covers most cases you will encounter.
- **Pitfall:** Incorrectly canceling terms instead of common factors (e.g., simplifying $\frac{x^2+1}{x}$ to $x + 1$). Always factor completely before canceling.

Worked Example: Resolving 0/0 by Factoring

Problem: Find the limit: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

Solution:

1. **Identify the Indeterminate Form:** Direct substitution gives $\frac{2^2 - 4}{2 - 2} = \frac{0}{0}$. This confirms we must do more work.
2. **Apply an Algebraic Technique:** In this case, we can **factor** the numerator, which is a difference of squares.
$$\lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$
3. **Simplify and Cancel:** Cancel the (x - 2) term from the numerator and denominator. $\lim_{x \to 2}(x + 2)$
4. **Re-evaluate the Limit:** Now, use direct substitution on the simplified function. 2 + 2 = 4

Answer: The limit is **4**.

Practice Exercises

Instructions: Find the value of each limit.

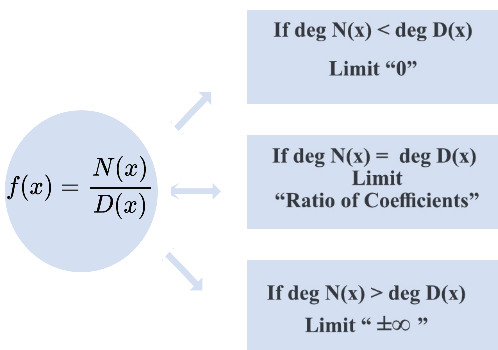
1. $\lim_{x \to -1} \frac{x^2 + 5x + 4}{x + 1}$	**3.** $\lim_{x \to 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$
2. $\lim_{x \to 0} \frac{\sqrt{x + 16} - 4}{x}$	**4.** $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

The Core Principle: The Battle of the Infinites

When direct substitution into a limit results in the form ∞/∞, this is our second major **indeterminate form**. It signifies a "battle" between the numerator and the denominator to see which one grows faster as x goes to infinity.

For rational functions (a polynomial divided by a polynomial), resolving this form is the exact same process as finding horizontal asymptotes: we **compare the degrees** of the numerator and denominator.

The Core Graphic: Quick Guide to End Behavior



- **Visual**: Three simple boxes:
  - **Box 1:** "Degree N < Degree D" → "Limit = 0"
  - **Box 2:** "Degree N = Degree D" → "Limit = Ratio of Coefficients"
  - **Box 3:** "Degree N > Degree D" → "Limit = ±∞"
- **Purpose**: This graphic serves as a quick-reference "cheat sheet" for the three possible outcomes when dealing with the ∞/∞ form in rational functions at infinity.

The Complete Toolkit: The Degree Comparison Rules

To find $\lim\limits_{x \to \mp\infty}$ for a rational function $\frac{ax^n + \cdots}{bx^m + \cdots}$

Degree Comparison	Limit Result	Horizontal Asymptote
n **<** ***m*** (Bottom-Heavy)	0	$y = 0$
n **=** ***m*** (Equal Degrees)	$\frac{a}{b}$ (Ratio of leading coefficients)	$y = \frac{a}{b}$
n **>** ***m*** (Top-Heavy)	∞ or −∞	None

Pitfalls & Pro-Tips

- **Critical Pitfall:** Assuming that ∞/∞ is always 1. It is an indeterminate form, and the actual limit depends entirely on the relative growth rates of the top and bottom functions.
- **Pro-Tip:** When the limit is as x→−∞, be extra careful with signs, especially with odd powers and square roots. Remember that for a negative $x, \sqrt{x^2} = |x| = -x$.
- **Pitfall:** Applying these specific degree rules to non-rational functions like $\frac{e^x}{x^2}$ .While the concept of "faster growing function wins" still applies, the simple degree comparison is only for polynomials.

Worked Example: The Degree Comparison Method

Problem: Find the limit: $\lim_{x\to\infty} \frac{10x^2 - 3x + 1}{5x^2 + 2}$

Solution:

1. **Identify Form:** As $x \to \infty$, both numerator and denominator approach ∞. This is an ∞/∞ indeterminate form.
2. **Compare Degrees:** The degree of the numerator $\left(10x^2\right)$ is 2. The degree of the denominator $\left(5x^2\right)$ is 2.
3. **Find Ratio:** Since the degrees are equal ($n = m$), the limit is the ratio of the leading coefficients. $\frac{10}{5} = 2$
4. **Conclusion:** The limit is 2. This means the function has a horizontal asymptote at $y = 2$.

Answer: The limit is **2**.

Practice Exercises

Instructions: Find the value of each limit.

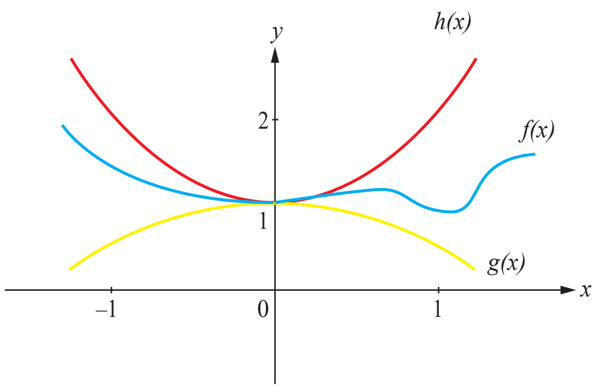
1. $\lim_{x\to\infty} \frac{x^2+5x-3}{x^3+2x}$	**3.** $\lim_{x\to\infty} \frac{\sqrt{9x^2+4}}{2x-1}$
2. $\lim_{x\to-\infty} \frac{4x^3+5x}{2x^2-7x}$	**4.** $\lim_{x\to-\infty} \frac{\sqrt{9x^2+4}}{2x-1}$

The Core Principle: Trapped Between Two Functions

The Squeeze Theorem is a powerful tool for finding the limit of a "difficult" function by trapping it between two "simpler" functions that both approach the same value.

The Idea: If a function $f(x)$ is always "squeezed" between two other functions, $g(x)$ and $h(x)$, and both of those outer functions approach the same limit L, then $f(x)$ has no choice but to also approach L. This is especially useful for limits involving oscillating functions like sine or cosine.

The Core Graphic: The Squeeze in Action



• **Visual**: Draw three functions: an upper function ***h*(*x*)**, a lower function ***g*(*x*)**, and a "squeezed" function ***f*(*x*)** that oscillates between them. Show that at a specific point $x = 0$, both $g(x)$ and $h(x)$ converge at the same limit point L, forcing $f(x)$ to converge there as well.

• **Purpose**: This graph provides the definitive intuitive proof for the Squeeze Theorem. If the outer functions come together, the function trapped in the middle must go with them.

The Toolkit: The Squeeze Theorem Method

To apply the Squeeze Theorem, follow these three steps:

1. **Bound It:** Start with the known bounds of the oscillating part of the function (e.g., $-1 \leq \sin(u) \leq 1$).
2. **Build It:** Algebraically manipulate all three parts of the inequality to make the middle part match the function in the limit problem.
3. **Squeeze It:** Find the limits of the two outer functions. If they are equal to the same value, L, then by the Squeeze Theorem, the limit of the middle function is also L.

Pitfalls & Pro-Tips

• **Critical Pitfall:** Trying to use the Product Rule for limits when one of the limits does not exist. The Squeeze Theorem is necessary because rules like the Product Rule fail for an oscillating function like $\sin(1/x)$.

• **Pro-Tip:** If you see a limit with sine or cosine of a complex argument (like $1/x$) multiplied by a term that goes to zero, your first thought should be "Squeeze Theorem".

• **Pitfall:** When building the inequality, forgetting to flip the inequality signs if you multiply by a negative number.

Worked Example: Squeezing an Oscillating Function

Problem: Find the limit: $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

Solution:

1. **Bound It:** We know the cosine function is always bounded between -1 and 1. $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$
2. **Build It:** Multiply all three parts of the inequality by x^2. Since x^2 is always non-negative, the inequality signs do not flip. $-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$
3. **Squeeze It:** Find the limits of the outer "squeezing" functions as $x \to 0$.
    - $\lim_{x\to 0}(-x^2) = 0$
    - $\lim_{x\to 0}(x^2) = 0$

   Since our target function is trapped between two functions that both approach 0, our function's limit must also be 0.

Answer: By the Squeeze Theorem, the limit is **0**.

Practice Exercises

Instructions: Use the Squeeze Theorem to find the value of each limit.

1. $\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right)$	**3.** If $2x \leq g(x) \leq x^4 - x^2 + 2x$ for all x, find $\lim_{x\to 1} g(x)$
2. $\lim_{x\to\infty} \frac{\sin(x)}{x}$	**4.** $\lim_{x\to 0^+} \sqrt{x} \cdot e^{\cos\left(\frac{1}{x}\right)}$

CHECKPOINT 1.3 – Indeterminate Mastery & Squeeze Use

Instructions: Answer the following questions to check your understanding of indeterminate forms (0/0, ∞/∞) and the Squeeze Theorem.

1. What is the value of the limit $\lim_{x \to -5} \frac{x^2 - 5}{x + 5}$?

(A) –10 (B) 0

(C) 10 (D) Does Not Exist

2. What is the value of the limit $\lim_{x \to \infty} \frac{1 - 3x^2}{x^2 + 4}$?

(A) –3 (B) $\frac{1}{4}$

(C) 1 (D) ∞

3. What is the value of the limit $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$?

(A) 0 (B) 1

(C) ∞ (D) Does Not Exist

4. What is the value of the limit $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$?

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$

(C) 0 (D) Does Not Exist

Answers & Feedback Loop

Answers and Explanations

1. (A) −10. Direct substitution gives 0/0.

Factoring the numerator gives $\frac{(x+5)(x-5)}{x+5}$.

Canceling the (x + 5) terms leaves $\lim_{x\to -5}(x-5)$,

which is −5 − 5 = −10.

2. (A) −3. This is an ∞/∞ form.

Since the degrees of the numerator $\left(x^2\right)$ and denominator $\left(x^2\right)$ are equal, the limit is the **ratio of the leading coefficients**: −3 / 1 = −3.

3. (A) 0. This requires the **Squeeze Theorem**.

We know $-1 \le \cos\left(\frac{1}{x}\right) \le 1$. Multiplying by x (for x near 0) gives $-|x| \le x \cdot \cos\left(\frac{1}{x}\right) \le |x|$.

Since $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0} |x| = 0$, the limit of the function squeezed between them must also be 0.

4. (A) 1/6. Direct substitution gives 0/0.

We must multiply the numerator and denominator by the **conjugate**,$(\sqrt{x}+3)$.

This gives $\lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}$.

After cancelling, we have

$$\lim_{x\to 9} \frac{1}{(\sqrt{x}+3)} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Self-Assessment Guide

All Correct: Excellent! You are skilled at handling the most common algebraic challenges with limits. You are ready to study continuity.

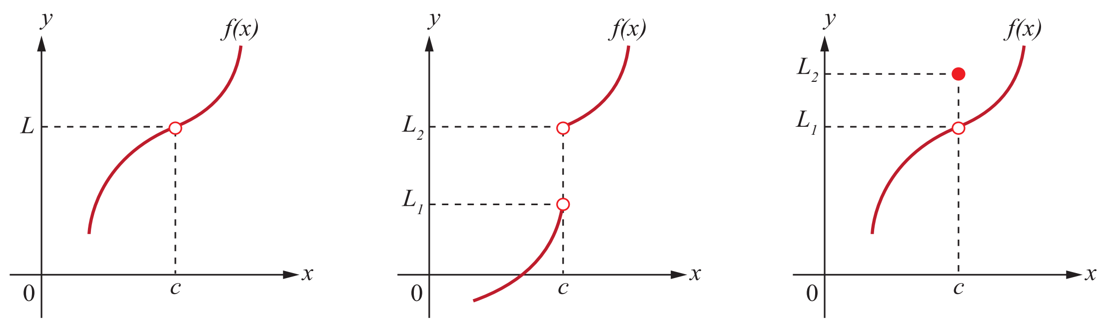
One Mistake: Very good work. Review the specific algebraic technique (factoring, conjugates) or theorem for the question you missed.

More Than One Mistake: Stop and Review. Mastery of these algebraic methods is absolutely essential. Please re-study topics 1.9 through 1.11 before proceeding to the next section.

The Core Principle: The "No-Lift" Rule

Intuitively, a function is **continuous** if you can draw its entire graph without ever lifting your pencil from the paper. This means the graph has no unexpected breaks, jumps, or holes. Formally, for a function to be continuous at a point, its expected value (the limit) must meet its actual value (the point).

The Core Graphic: The Continuity Checklist



- **Visual 1 (Fails #1):** The point $f(c)$ is not defined.
- **Visual 2 (Fails #2):** The limit does not exist (left ≠ right).
- **Visual 3 (Fails #3):** The limit exists, but does not equal the point's value.
- **Purpose:** This visual set clearly illustrates how a function can fail each of the three conditions, making the formal definition concrete.

The Toolkit: The Three-Condition Checklist

For a function $f(x)$ to be continuous at a point $x=c$, it must satisfy **all three** of the following conditions. If even one fails, the function is discontinuous at c.

1. **The Point Exists:** $f(c)$ must be defined.
2. **The Limit Exists:** $\lim_{x \to c} f(x)$ must exist (meaning the left- and right-hand limits are equal).
3. **The Values Match:** $\lim_{x \to c} f(x) = f(c)$

Pitfalls & Pro-Tips

• **Critical Pitfall:** Only checking if the limit exists. A limit can exist at a hole in the graph, but for continuity, the function's point must plug that hole perfectly.

• **Pro-Tip:** Most functions you know—polynomials, sine, cosine, and exponential functions—are continuous everywhere. Rational functions are only discontinuous where their denominator is zero.

• **Pitfall:** Forgetting that a function must be defined at the point. If $f(c)$ is undefined, the function is automatically discontinuous at c.

Worked Example: Applying the Three-Part Test

Problem: Is the piecewise function $g(x) = \begin{cases} x+2, x \neq 3 \\ 1, x = 3 \end{cases}$ continuous at $x = 3$?

Solution: We check the three conditions methodically:

1. **Is $g(3)$ defined?** Yes. According to the second piece of the function, $g(3) = 1$. Condition 1 passes.
2. **Does the limit exist at $x = 3$?** Yes. For the limit, we use the $x \neq 3$ piece:
   $\lim_{x\to 3}(x+2) = 3+2 = 5$. Condition 2 passes.
3. **Does the limit equal the function value?** We compare the results. Does $\lim_{x\to 3} g(x) = g(3)$?
   Does $5 = 1$? No. Condition 3 fails.

Answer: Since one of the conditions fails, the function is **not continuous** at $x = 3$.

Practice Exercises

Instructions: Use the three-part definition of continuity to determine if the function is continuous at the specified point. State which condition fails, if any.

1. Is the function $f(x) = \frac{x^2-9}{x-3}$ continuous at $x = 3$?

2. Is $k(x) = \begin{cases} 2x+3, x < -1 \\ x, x \geq -1 \end{cases}$ continuous at $x = -1$?

3. Find the value of the constant c that makes the function $f(x) = \begin{cases} x^2 - c, & x < 5 \\ cx + 6, & x \geq 5 \end{cases}$ continuous everywhere.

4. If $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 9$, find all x-values where the composite function $h(x) = f(g(x))$ is discontinuous.

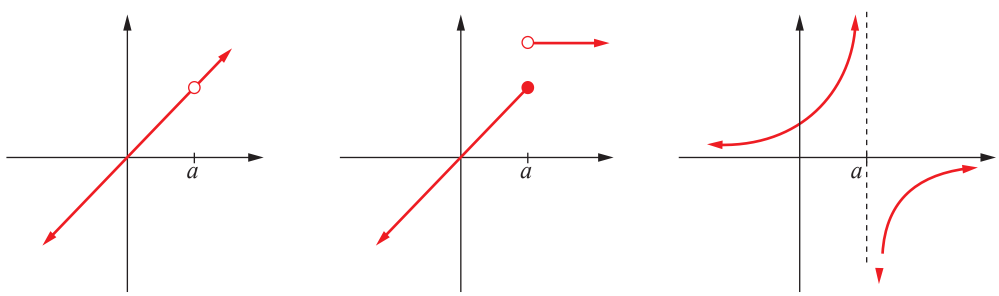
The Core Principle: Can It Be Fixed?

When a function fails the continuity test at a point, we classify the discontinuity into two main types based on a simple question: Can we easily "fix" the break?

1. Removable Discontinuity (Holes): This occurs when the limit **exists**, but is not equal to the function's value (or the function is not defined at the point). It's a "hole" in the graph that we could "plug" by redefining a single point.

2. Non-Removable Discontinuity (Jumps & Asymptotes): This occurs when the limit **does not exist**. This is a more severe break that cannot be fixed by a single point.

The Core Graphic: Giving a Name to the Problem



- **Visual 1 (The Hole):** Label it clearly: **Removable Discontinuity**.
- **Visual 2 (The Jump):** Label it clearly: **Non-Removable Discontinuity (Jump)**.
- **Visual 3 (The Asymptote):** Label it clearly: **Non-Removable Discontinuity (Infinite)**.
- **Purpose:** This visual directly links the *reason* a function is discontinuous to the *name* of the discontinuity type.

The Toolkit: Classifying Discontinuities

Discontinuity Type	Limit Result	How it Fails the Continuity Test
Removable (Hole)	Exists	Condition #1 or #3 fails (point undefined or limit $\neq$ point).
Non-Removable (Jump)	Does Not Exist (LHL $\neq$ RHL)	Condition #2 fails.
Non-Removable (Infinite)	Does Not Exist (approaches $\pm\infty$)	Condition #2 fails (and also #1).

Pitfalls & Pro-Tips

• **Critical Pitfall:** Assuming any value making the denominator zero is a vertical asymptote. You **must factor and simplify first!** If a factor in the denominator **cancels**, it creates a **hole** (removable). If the factor **remains**, it creates a **vertical asymptote** (non-removable).

• **Pro-Tip:** The names tell the story. "Removable" means you can remove the break. "Non-removable" means you can't. Jumps and asymptotes are non-removable breaks.

• **Pitfall:** Not checking both sides of a piecewise function at a break point to see if it's a jump.

Worked Example: Finding Holes and Asymptotes

Problem: Find and classify the discontinuities for the function $f(x) = \dfrac{x^2-4}{x^2-5x+6}$

Solution:

1. **Factor the function completely.** $f(x) = \dfrac{(x-2)(x+2)}{(x-2)(x-3)}$

2. **Identify Canceled Factors.** The $(x-2)$ term appears in both the numerator and the denominator, so it cancels. This indicates a **removable discontinuity (a hole)** at $x = 2$.

3. **Identify Remaining Factors in Denominator.** The $(x-3)$ term remains in the denominator. This indicates a **non-removable, infinite discontinuity (a vertical asymptote)** at $x = 3$.

4. **(Bonus) Find the coordinates of the hole:** Use the simplified function, $f(x) = \dfrac{x+2}{x-2}$, and plug in The x-value of the hole, $x = 2$. $y = \dfrac{2+2}{2-3} = \dfrac{4}{-1} = -4$. The hole is at (2, 4).

Answer: Removable discontinuity at $x = 2$; Non-removable (infinite) discontinuity at $x = 3$.

Practice Exercises

Instructions: For each function, find the x-values of any discontinuities and classify them.

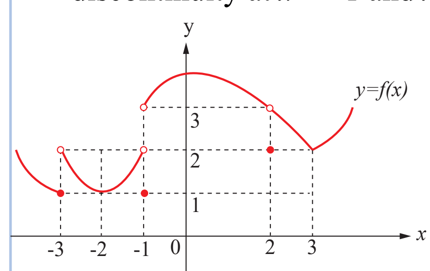
1. $f(x) = \dfrac{x+1}{x^2-1}$

3. $g(x) = \dfrac{x^2-x-6}{x^2+x-2}$

2. Find and classify the discontinuity for

$$h(x) = \begin{cases} x^2, x < 2 \\ 5, x \geq 2 \end{cases} \text{ at } x = 2.$$

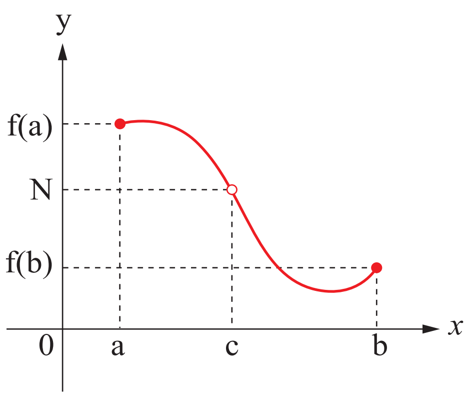
4. For the function graphed below, classify the discontinuity at $x = -1$ and $x = 2$.



The Core Principle: You Can't Skip Values

The Intermediate Value Theorem (IVT) provides a powerful guarantee for **continuous functions**. The intuitive idea is simple: if you are driving on a road and your journey is continuous (you don't teleport!), you must pass through every single elevation between your starting and ending points. In calculus, we use this primarily to prove that a **root** (a zero) exists for a function on a given interval.

The Core Graphic: Guaranteeing an Intersection



- **Visual**: Draw a **continuous** function on a closed interval $[a, b]$. Mark the endpoints $(a, f(a))$ and $(b, f(b))$. Pick an "intermediate" y-value, N, between $f(a)$ and $f(b)$. Draw a horizontal line from $y = N$ across to the function, showing that it **must** intersect the graph at some x-value, c.
- **Purpose**: This visual proves that for a continuous path between two y-values, any intermediate y-value N must be crossed at least once.

The Toolkit: The IVT Conditions and Guarantee

For the Intermediate Value Theorem to apply, two conditions **must** be met:

1. The function $f(x)$ is **continuous** on the closed interval $[a, b]$.
2. N is any y-value strictly between $f(a)$ and $f(b)$.

The IVT Guarantee: If the conditions are met, then there is **at least one** x-value c in the interval (a, b) such that $f(c) = N$.

Pitfalls & Pro-Tips

- **Critical Pitfall:** Forgetting to state that the function is **continuous** in your justification. This is the most important condition, and omitting it will lose points on an exam.
- **Pro-Tip:** The IVT is an **existence theorem**. It tells you a value c *exists*, but it does not tell you *how to find* its exact value.
- **Pitfall:** Attempting to apply the IVT to a function that is not continuous on the entire interval. If there's a discontinuity, the theorem is not applicable.

Worked Example: Justifying the Existence of a Root

Problem: Use the IVT to show that $f(x) = x^3 + 2x - 5$ has a root on the interval [1, 2].

Solution: We follow a three-step justification:

1. Check for Continuity: State that $f(x)$ is a polynomial, which is continuous everywhere. Therefore, it is continuous on the closed interval [1, 2].

2. Evaluate Endpoints: Calculate the y-values at the endpoints of the interval.

- $f(1) = (1)^3 + 2(1) - 5 = -2$
- $f(2) = (2)^3 + 2(2) - 5 = 7$

3. Apply IVT: We are looking for a root, which means our target intermediate value is $N = 0$.

Since $f(x)$ is continuous and $N = 0$ is between the endpoint values $f(1) = -2$ and $f(2) = 7$, the IVT guarantees a solution exists.

Answer: By the IVT, a root is guaranteed to exist on the interval [1, 2].

Practice Exercises

Instructions: Use the Intermediate Value Theorem to answer the following questions.

1. Use the IVT to show that $f(x) = x^3 - 4x - 2$ has a root on the interval [2, 3].

2. Can the IVT be used to claim there is a value c in the interval [3, 5] such that $g(c) = 0$ for the function $g(x) = \frac{1}{x-4}$? Explain why or why not.

3. The function h (t) is continuous. Selected values are given in the table below. On which interval is there guaranteed to be a time t such that h (t) = 10?

t	0	5	10	15
$h(t)$	–2	8	12	20

4. Use the IVT to prove that the graphs of the functions $f(x) = \cos(x)$ and $g(x) = x$ must intersect. (Hint: Create a new function $h(x) = f(x) - g(x)$ and test for a root)

CHECKPOINT 1.4 – Continuity & IVT

Instructions: Instructions: Answer the following questions to check your understanding of the definition of continuity, types of discontinuities, and the Intermediate Value Theorem.

1. For what value of the constant c is the function $f(x)$ continuous at $x = 2$?

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & ,x \neq 2 \\ c & ,x = 2 \end{cases}$$

(A) 0 (B) 2

(C) 4 (D) There is no such value of c.

2. How would you classify the discontinuity at $x = 1$ for the function $g(x) = \frac{x+2}{x-1}$?

(A) Removable (Hole)

(B) Non-removable (Jump)

(C) Non-removable (Infinite)

(D) The function is continuous at $x = 1$.

3. The function $h(x)$ is continuous on the interval [1, 5]. The table below shows some values for $h(x)$. The Intermediate Value Theorem guarantees that the equation $h(x) = 0$ has a solution on which of the following intervals?

x	1	2	3	4	5
h(x)	–4	–1	2	3	1

(A) [1, 2] (B) [2, 3]

(C) [3, 4] (D) [4, 5]

4. Which of the following conditions is not required for the Intermediate Value Theorem to apply on an interval [a, b]?

(A) The function must be continuous on $[a, b]$.

(B) The limit must exist at every point in (a, b).

(C) The function must be defined at x = a and $x = b$.

(D) The function must be differentiable on (a, b).

Answers & Feedback Loop

Answers and Explanations

1. (C) 4

For the function to be continuous, the limit must equal the point's value. First, find the limit:

$$\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

So, we must define c to be 4 to "plug the hole."

2. (C) Non-removable (Infinite). Direct substitution at $x = 1$ gives 3/0, which indicates a **vertical asymptote**. A vertical asymptote is a type of non-removable, infinite discontinuity.

3. (B) [2, 3]. The IVT guarantees a root (where $h(x) = 0$) when the function values at the endpoints of an interval have opposite signs. On the interval [2, 3], the function goes from $h(2) = -1$ (negative) to $h(3) = 2$ (positive), so it must cross 0 in between.

4. (D) The function must be differentiable on (a, b). The IVT only requires the function to be **continuous** on the closed interval. Differentiability is a stronger condition and is not required for the IVT to apply.

Self-Assessment Guide

All Correct: Excellent! You have a solid understanding of continuity and the IVT. You are ready for the BC-only topics.

One Mistake: Good job. Review the specific definition or theorem for the question you missed. A quick refresher should be all you need.

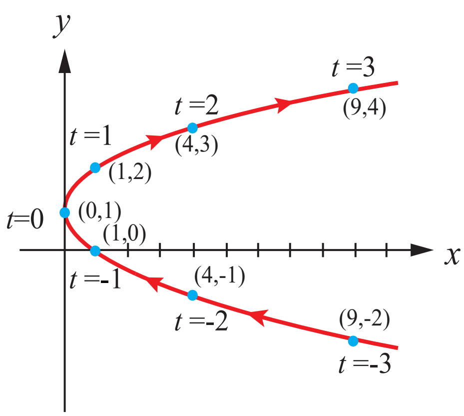
More Than One Mistake: Stop and Review. Continuity is a gateway concept for the rest of calculus. It's critical to re-study topics 1.12 through 1.14 before moving on.

The Core Principle: A Limit with Coordinates

Finding the limit of a parametrically defined curve, $(x(t),y(t))$, is a straightforward extension of standard limits. To find where the curve approaches as the parameter t approaches a value k, we simply find the limits of the x and y components separately. The result is not a single number, but a **coordinate point** (X, Y) that the curve approaches on the Cartesian plane.

The Core Graphic: A Point's Journey on a Curve

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



• **Visual**: The graphic consists of two parts: a table of values on the left and its corresponding graph on the right. The graph plots the (x,y) points generated by the parameter t, forming a sideways parabola.

• **Labels:** Key points on the curve are labeled with their corresponding t-values (e.g., $t = 2$ at point (4,3)). Arrows along the curve indicate the direction of motion as the parameter t increases from -3 to 3.

• **Purpose**: This combined visual powerfully demonstrates how a parametric curve is traced as the parameter t changes. It allows a student to see both the numerical relationship in the table and the resulting geometric path on the graph, making the concept of a limit at a specific t-value (approaching a specific point on the curve) intuitive.

The Toolkit: The Formal Process for Parametric Limits

The Rule: If a curve is defined by $(x(t), y(t))$, then: $\lim_{t\to k}\left(x(t), y(t)\right) = \left(\lim_{t\to k}\left(x(t)\right), \lim_{t\to k}\left(y(t)\right)\right)$

The Process:

1. Calculate the limit of the x-component using all known limit techniques.
2. Calculate the limit of the y-component using all known limit techniques.
3. Combine the results into an ordered pair (X, Y).

Pitfalls & Pro-Tips

• **Critical Pitfall:** Giving a single number as the answer. The limit of a parametric curve is a **point**, which must be written as a coordinate pair (X, Y).

• **Pro-Tip:** All the limit techniques you have learned (Direct Substitution, Factoring, Degree Comparison, Squeeze Theorem) can be applied to each component individually.

• **Pitfall:** If either the x-component's limit or the y-component's limit does not exist or is infinite, the overall coordinate limit **Does Not Exist**.

Worked Example: Finding a Coordinate Limit

Problem: Find the limit as $t \to 0$ for the curve defined by $x(t) = t^2 + 1$ and $y(t) = \frac{\sin(t)}{t}$.

Solution: We evaluate the limit for each component separately.

1. **Find the limit of the *x*-component:** $\lim_{t\to 0} x(t) = \lim_{t\to 0}\left(t^2+1\right)$.
   Using direct substitution: $(0)^2 + 1 = 1$.
2. **Find the limit of the *y*-component:** $\lim_{t\to 0} y(t) = \lim_{t\to 0}\left(\frac{\sin(t)}{t}\right)$.
   This is a special trigonometric limit that equals 1.
3. **Combine the results into a coordinate:** The limit of the curve is the point (lim $x(t)$, lim $y(t)$).

Answer: As $t \to 0$, the position approaches the point **(1, 1)**.

Practice Exercises

Instructions: Find the limit of the curve defined by the given parametric equations as t approaches the specified value.

1. Find the limit as $t \to 2$ for $x(t) = 3t - 5$ and $y(t) = t^2 + 4$.

2. Find the limit as $t \to 1$ for $x(t) = \frac{t^2-1}{t-1}$ and $y(t) = \frac{t+2}{t+1}$

3. Find the end behavior (limit as $t \to \infty$) for the curve defined by $x(t) = \frac{4t^2}{t^2+1}$ and $y(t) = \frac{3}{t}$.

4. Find the limit as $t \to 0$ for the curve defined by $x(t) = \cos(t)$ and $y(t) = \frac{1}{t^2}$

The Core Principle: The Limit is the Equation

This special BC topic covers limits of infinitely nested expressions, like $\sqrt{6+\sqrt{6+\cdots}}$ These problems look intimidating, but they can be solved with a clever algebraic trick. The core idea is this: If an infinitely repeating expression converges to a limit ***L***, then the "inner" part of the expression, which is identical to the whole thing, must also be converging to ***L***. This allows us to create a simple, self-referencing equation to solve for L.

The Core Graphic: The Infinite Substitution

$$\underbrace{\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}_{L} = \sqrt{6+\underbrace{\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}_{L}}$$

- **Visual:** 1. Show the full expression: $\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$
  2. Circle the entire expression and label it **"=** ***L*****"**.
  3. Circle only the *inner* part (everything after the first '6+'). Draw an arrow showing that this is also identical to the whole expression, and thus can be replaced with ***L***.
  4. This leads to the visual equation: $L=\sqrt{6+L}$
- **Purpose:** This visual makes the abstract concept of recursive substitution concrete.

The Complete Toolkit: The Self-Referencing Equation Method

1. **Assume & Name:** Assume the limit exists and set the entire expression equal to a variable, like L.
2. **Identify the Pattern:** Notice that because the expression is infinite, the part "inside" the first operation is identical to the whole expression.
3. **Create an Equation:** Substitute L back into the expression to form a simple algebraic equation.
4. **Solve for *L*:** Solve the equation, which is usually a quadratic.
5. **Check & Conclude:** Discard any extraneous solutions (e.g., negative answers for a positive root).

Pitfalls & Pro-Tips

- **Critical Pitfall:** Forgetting to check for extraneous solutions. After solving your algebraic equation, always look back at the original problem to see if your answer makes sense. A limit of positive square roots cannot be negative.
- **Pro-Tip:** This technique is not limited to radicals. It can also be applied to other nested expressions, like infinite continued fractions.
- **Assumption:** These problems almost always work under the assumption that the sequence converges to a finite limit.

Worked Example: Solving the Infinite Puzzle

Problem: Find the limit: Find the value of: $\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}$

Solution:

Name the limit: Let $L = \sqrt{6+\sqrt{6+\dots}}$

1. **Create the equation:** The expression inside the first square root is identical to : $L = \sqrt{6+L}$
2. **Solve the equation:**
   - Square both sides: $L^2 = 6 + L$
   - Rearrange: $L^2 - L - 6 = 0$
   - Factor: $(L-3)\cdot(L+2) = 0$ This gives two potential solutions: $L = 3$ and $L = -2$.
3. **Check and Conclude:** The original expression involves only positive square roots, so the limit must be positive. We discard the extraneous solution $L = -2$.

Answer: The limit is **3**.

Practice Exercises

Instructions: Assuming the limit exists, find the value of each infinitely nested expression.

1. Find the value of: $\sqrt{30+\sqrt{30+\sqrt{30+\dots}}}$

3. Find the value of the infinite continued fraction:

$$2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}$$

2. Find the value of: $\sqrt{42-\sqrt{42-\sqrt{42-\cdots}}}$

4. Find the value of: $\sqrt{1+2\sqrt{1+2\sqrt{1+2\dots}}}$

CHECKPOINT 1.5 – Parametric & Radical Limits

Instructions: Answer the following questions to check your understanding of the BC-only limit topics: parametric limits and nested radical limits.

1. A curve is defined by the parametric equations

$$x(t) = t^2 + 3 \text{ and } y(t) = \frac{t}{2}.$$

What is the coordinate point the curve approaches as $t \to 3$?

(A) (9, 3/2)   (B) (12, 3/2)

(C) (9, 1)   (D) Does Not Exist

2. What is the value of the infinitely nested radical $\sqrt{20+\sqrt{20+\sqrt{20+\ldots}}}$?

(A) 4   (B) 5   (C) 10   (D) 20

3. A curve's end behavior is described by

$$x(t) = \frac{1-t^2}{t^2+1} \text{ and } y(t) = 5e^{-t}.$$

What point does the curve approach as $t \to \infty$?

(A) (-1, 5)   (B) (1, 0)

(C) (-1, 0)   (D) Does Not Exist

4. What is the value of the infinite continued fraction

$$6 - \cfrac{5}{6 - \cfrac{5}{6 - \ldots}} \ ?$$

(A) 1   (B) 4   (C) 5   (D) 6

Answers & Feedback Loop

Answers and Explanations

1. (B) (12, 3/2)

We find the limit of each component using direct substitution:

- $\lim_{t\to 3} x(t) = (3)^2 + 3 = 12$
- $\lim_{t\to 3} y(t) = 3/2$ The coordinate limit is (12, 3/2).

2. (B) 5

Let the limit be L. The equation is $L = \sqrt{20 + L}$.
Squaring both sides gives $L^2 = 20 + L$, or $L^2 - L - 20 = 0$. Factoring gives (L − 5)(L + 4) = 0. Since the limit must be , Positive we discard L = – 4. The answer is 5.

3. (C) (– 1, 0)

We find the limit of each component as t → ∞:

- $\lim_{t\to \infty} x(t) = \frac{-1}{1} = -1$ (by comparing degrees).
- $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{5}{e^t} = 0$. The coordinate limit is (– 1, 0).

4. (C) 5

Let the limit be L. The equation is $L = 6 - \frac{5}{L}$.
Multiplying by L gives $L^2 = 6L - 5$, or $L^2 - 6L + 5 = 0$ Factoring gives $(L - 5)(L - 1) = 0$.
The solutions are $L = 5$ and $L = 1$. Looking at the sequence, the first term is 6, the second is 6 − 5/6 ≈ 5.17. The sequence appears to approach 5 from above, so $L = 5$ is the correct limit.

Self-Assessment Guide

All Correct: Excellent! You have mastered the BC-specific limit topics for this unit. You are fully prepared for the Unit 1 Test.

One Mistake: Good work. Briefly review the topic for the question you missed. These are unique problem types, so a quick refresher is helpful.

More Than One Mistake: Stop and Review. These BC topics often reappear in later units. Ensure you are comfortable with the algebraic process for both parametric and recursive limits before moving on.

1. What is the value of $\lim_{x\to 2}\left(x^2-3x+5\right)=$?

(A) –7 (B) 3

(C) 15 (D) Does Not Exist

2. If and $\lim_{x\to 1} f(x)=3$ and $\lim_{x\to 1} g(x)=-2$

What is $\lim_{x\to 1}[2f(x)-g(x)]=$?

(A) 1 (B) 4

(C) 5 (D) 8

3. What is the value of $\lim_{x\to 5^+}\frac{1}{x-5}=$?

(A) $-\infty$ (B) 0

(C) 1 (D) $+\infty$

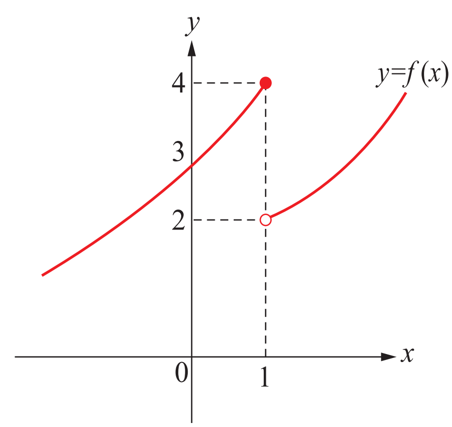
4. Given $f(x)=\begin{cases} x+2, x<1 \\ 4, x\geq 1\end{cases}$

What $\lim_{x\to -2} f(x)=$?

(A) 0 (B) 3

(C) 4 (D) Does Not Exist

5.



Based on the graph of the function $f(x)$ shown above, what is the value of $\lim_{x\to 1} f(x)=$?

(A) 1 (B) 2

(C) 4 (D) Does Not Exist

6. What is the value of $\lim_{x\to 1}|x-10|=$?

(A) –9 (B) 11

(C) 9 (D) Does Not Exist

7. What is the value of $\lim_{x\to\infty} \frac{5x+1}{x^2-3} = ?$

(A) $\frac{1}{3}$ (B) 0 (C) 5 (D) ∞

8. What is the value of $\lim_{x\to 1} \frac{x^2-1}{x+1} = ?$

(A) 0 (B) 1 (C) 2 (D) Does Not Exist

9. What is the value of $\lim_{x\to 1} \frac{x^2-1}{x-1} = \ ?$

(A) 2 (B) 4 (C) ∞ (D) 0

10. What is the value of $\lim_{x\to 0} x \cdot \cos\left(\frac{1}{x}\right) = ?$

(A) 0 (B) 1 (C) ∞ (D) Does Not Exist

11. Is the function $f(x) = 3x - 1$ continuous at $x = 5$?

(A) Yes, because $\lim f(x)$ exists.

(B) Yes, because $f(5)$ is defined and equals the limit.

(C) No, because $f(5)$ is not defined.

(D) No, because the limit does not exist.

12. What type of discontinuity does $f(x) = \frac{x+2}{x+2}$ have at $x = -2$

(A) Jump

(B) Infinite

(C) Removable

(D) The function is continuous

1. If $\lim_{x\to 2}\frac{f(x)}{x-1}=5$, what is $\lim_{x\to 2} f(x)=?$

(A) 1 (B) 5

(C) 10 (D) Cannot be determined

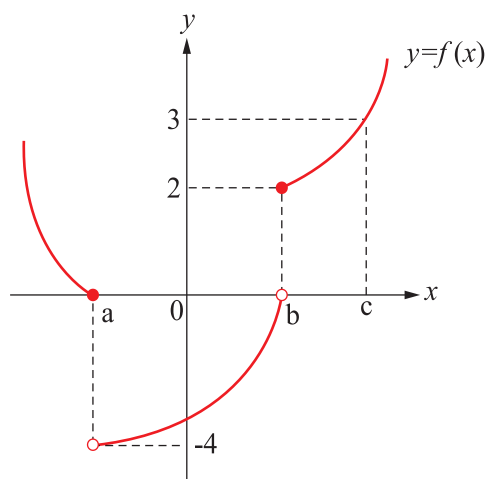
2. What is the value of $\lim_{x\to 1^-}\frac{x+2}{x-1}=?$

(A) $-\infty$ (B) $+\infty$ (C) 3 (D) Does Not Exist

3. Given $f(x)=\begin{cases} x^2+1, x<2 \\ 6-x, x\geq 2\end{cases}$ What $\lim_{x\to -2} f(x) = ?$

(A) 4 (B) 5 (C) 6 (D) Does Not Exist

4.



The graph of the function $y=f(x)$ is shown above. Based on the graph, what is the value of the following expression?

$$\lim_{x\to a^+} f(x)+\lim_{x\to b^-} f(x)+\lim_{x\to c^+} f(x)$$

A) -1 B) 1 C) 3 D) 5

5. What is the value of $\lim_{x\to 5}\frac{x-5}{|x-5|}=?$

(A) -1 (B) 1 (C) 0 (D) Does Not Exist

6. What is the value of $\lim_{x\to\infty}\frac{x^3+2x}{1-x^2}=?$

(A) $-\infty$ (B) -1 (C) 0 (D) $+\infty$

7. What is the value of $\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$ = ?

(A) 1/6 (B) 1/3
(C) 0 (D) Does Not Exist

8. What is the value of $\lim_{x\to \infty}\frac{\sqrt{16x^2+5x}}{2x-1}$ = ?

(A) 2 (B) 4 (C) 8 (D) ∞

9. For what value of the constant k is the function

$$f(x)=\begin{cases} kx+1, & x\le 3 \\ 2x-k, & x>3 \end{cases}$$ continuous at $x = 3$?

(A) 1 (B) 5/4 (C) 2 (D) 5/2

10. What type of discontinuity does $f(x)=\frac{x}{|x|}$ have at $x = 0$?

(A) Removable
(B) Jump
(C) infinite
(D) Oscillating

11. Given $f(x)=x^3-3x+1,$ the IVT guarantees a root on which interval?

(A) [−4, −3] (B) [−1, 0]
(C) [0, 1] (D) [2, 3]

12. What is the value of $\lim_{x\to \pi}\frac{sin(2x+sin(x))}{x}$ = ?

(A) −1 (B) 0
(C) 1 (D) Does Not Exist

1. What is the value of the limit $\lim_{x\to 0}\dfrac{\frac{1}{x+2}-\frac{1}{2}}{x}$?

(A) –1/4 (B) 1/4

(C) 4 (D) Does Not Exist

2. What is the value of the limit $\lim_{x\to -\infty}\dfrac{3x-2}{\sqrt{4x^2+1}}$?

(A) –3/2 (B) 3/4

(C) 3/2 (D) ∞

3. If a function $f(x)$ is not continuous on the interval [a, b], what can be concluded about the existence of a value ***c*** in (a, b) such that $f(c)=N$ for an intermediate value N?

(A) The IVT guarantees a value c.

(B) The IVT guarantees there is no value c.

(C) The IVT does not apply and makes no guarantee.

(D) The function must have a vertical asymptote.

4.

$$f(x)=\begin{cases} ax+b, x<1 \\ 5, x=1 \\ 2ax-b, x>1 \end{cases}$$

For what values of the constants ***a*** and ***b*** is the function below continuous for all real numbers?

(A) $a=3$, $b=2$ (B) $a=2$, $b=3$

(C) $a=10/3$, $b=5/3$ (D) $a=0$, $b=5$

5. The graph of $y=f(x)$ is defined as $f(x)=|x-2|$.

What is the value of $\lim_{x\to 0} f\left(x^2-2\right)$?

(A) 0 (B) 2

(C) 4 (D) Does Not Exist

6. What is the value of the limit $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}$?

(A) 0 (B) 4 (C) 12 (D) ∞

7. For which of the following functions does $\lim_{x \to 4} f(x)$ exist?

I. $f(x) = |x - 4|$

II. $f(x) = \dfrac{x^2 - 16}{x - 4}$

III. $f(x) = \begin{cases} 2x - 1, & x \leq 4 \\ x + 3, & x > 4 \end{cases}$

(A) I only
(B) II only
(C) I and II only
(D) I, II, and III

8. The function $f(x) = \dfrac{sin(x)}{x}$ has a discontinuity at $x = 0$. How is this discontinuity classified?

(A) Removable
(B) Non-removable (Jump)
(C) Non-removable (Infinite)
(D) The function is continuous at $x = 0$

9. Let g be a continuous function. If $\lim_{x \to \infty} g(x) = 4$, what is $\lim_{x \to \infty} g\left(e^{-x} + 4\right)$?

(A) 0
(B) 4
(C) g(4)
(D) Does Not Exist

10. If $e^x \leq f(x) \leq e^x + x^2$ for all $x > 0$, what is $\lim_{x \to 0^+} f(x)$?

(A) 0
(B) 1
(C) e
(D) Cannot be determined

11. Find the limit: $\lim_{x \to 2^-} \dfrac{x^2 - 4}{|x^2 - 5x + 6|}$

(A) –4
(B) 1
(C) 4
(D) Does Not Exist

12. If f is a continuous function on [0, 1] such that $f(0) = 1$ and $f(1) = 0$, which of the following statements is guaranteed by the Intermediate Value Theorem?

(A) $f(c) = c$ for some c in [0, 1].
(B) $f(c) = 1/2$ for some c in [0, 1].
(C) $f(c) = 2$ for some c in [0, 1].
(D) The function has a maximum value at $x = 0$.

1. A curve is defined by the parametric equations

$x(t)=\dfrac{t^2-9}{t-3}$ and $y(t)=2t+1$.

What is the position of the point on the curve as $t\to 3$?

(A) (0, 7) (B) (6, 7)

(C) (1, 7) (D) Does Not Exist

2. What is the value of the infinitely nested radical

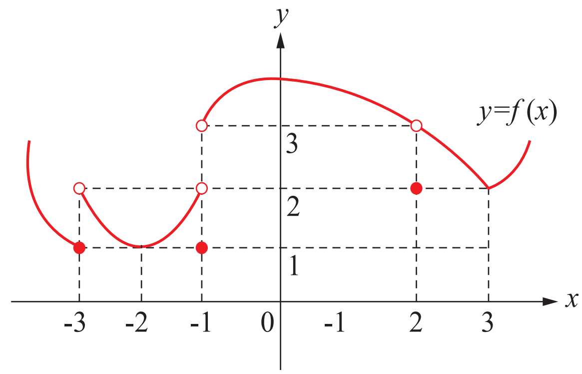
$\sqrt{42+\sqrt{42+\sqrt{42+\ldots}}}$?

(A) 6 (B) 7 (C) 35 (D) 42

3. What is the value of the limit $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+4x}}{x}$?

(A) -1 (B) 0 (C) 1 (D) ∞

4.



The graph of $y=f(x)$ is shown below. What is

$\lim\limits_{x\to-1^-} f(f(x))$?

(A) 1 (B) 2

(C) 3 (D) Does Not Exist

5. A curve's end behavior is described by the parametric equations $x(t)=4e^{-t}$ and $y(t)=\dfrac{1-2t}{t+1}$.

What is the position of the point as $t\to\infty$?

(A) (4, −2) (B) (0, −2)

(C) (∞, −2) (D) (0, 1)

6. What is the value of the limit $\lim\limits_{x\to\pi}\dfrac{\sin(x)}{x-\pi}$?

(A) −1 (B) 0

(C) 1 (D) Does Not Exist

7. A curve is defined by $x(t) = \frac{1}{t^2}$ and $y(t) = \cos(t)$.

What is the limit of the curve's position as $t \to 0$?

(A) (0, 1) (B) (∞, 1)

(C) (0, ∞) (D) Does Not Exist

8. What is the value of the continued fraction

$$4 + \cfrac{3}{4 + \cfrac{3}{4 + \dots}} \; ?$$

(A) 5 (B) $2 + \sqrt{7}$ (C) 4 (D) 3

9. The function f is continuous on [1,4] and has values shown below. Which of the following statements is guaranteed by the IVT?

x	1	2	3	4
$f(x)$	10	0	−5	−8

(A) $f(c) = -6$ for some c in [3, 4].

(B) $f(c) = 6$ for some c in [1, 2].

(C) There is a root on the interval [1, 2].

(D) All of the above are guaranteed.

10. What is the value of the limit $\lim_{x \to 0^+} \sqrt{x} \cdot \cos\left(\frac{1}{x^2}\right)$?

(A) 0 (B) 1

(C) ∞ (D) Does Not Exist

11. Find the limit: $\lim_{x \to 2^-} \frac{x^2 - 4}{|x^2 - 3x + 2|}$?

(A) −4 (B) 1

(C) 4 (D) Does Not Exist

12. A function is defined as

$$f(x) = \begin{cases} \sin(\pi x) & , x < 1 \\ k & , x = 1 \\ x^2 - 1 & , x > 1 \end{cases}$$

For the limit of $f(x)$ to exist as x approaches 1, what must be the value of k?

(A) −1

(B) 0

(C) 1

(D) The limit exists regardless of the value of k.

Unit 1 Summary – The 5W1H Box (Limits & Continuity)		
QUESTION	**EXPLANATION**	**KEY IDEAS**
WHAT	What core concepts did we learn in this unit?	• **Limit:** The behavior of a function near a point. • **Continuity:** No breaks, jumps, or holes. • **Intermediate Value Theorem (IVT):** A continuous function must take on all values between two points. • **Infinite & End Behavior Limits:** Vertical and horizontal asymptotes.
WHY	Why does this matter?	• **Foundation of Calculus:** Limits define both derivative and integral. • **Understanding Function Behavior:** Detect holes, jumps, and asymptotes. • **Proving Existence:** Use IVT to guarantee solutions within an interval.
HOW	How do we solve problems in this unit?	• **Limit Evaluation:** Direct substitution → factoring → conjugates → squeeze theorem → L'Hôpital's Rule (BC). • **Continuity Test:** 1. Defined? 2. Limit exists? 3. Equal? • **Asymptotes:** Check denominator zeros (vertical) and end behavior (horizontal).
WHEN	When do we use each method?	• Always start with direct substitution. • If result is 0/0 or ∞/∞ move to factoring or L'Hôpital. • Use IVT to prove a root exists in a continuous interval. • At break points, check one-sided limits.
WHERE	Where do these ideas show up?	• Limits near a specific point. • Continuity at a point or across an interval. • Horizontal asymptotes as $x \to \pm\infty$. • Discontinuities piecewise junctions or denominator zeros.
WHO	Which functions did we analyze?	• **Continuous:** polynomials, trigonometric, exponential. • **Discontinuous:** piecewise, absolute value, rational.

Calculator Mastery Hub

Your Strategic Guide to Graphing on the Digital AP Calculus Exam

Why This Section Matters

The AP Calculus exam is now administered in a digital format, offering students access to both the built-in Desmos Graphing Calculator and traditional handheld devices like the TI-84. While the use of Desmos is not mandatory, it is rapidly becoming the preferred tool due to its powerful visual and analytical capabilities.

Yet, many students are unfamiliar with how to apply Desmos under test conditions—where time, clarity, and accuracy matter most. This section bridges that gap.

What You Will Learn

The Calculator Mastery Hub teaches 12 essential graphing skills, each carefully chosen to reflect real AP-style questions where graphing tools provide a clear strategic edge. Each skill includes:

- A concise explanation of the concept
- A test-ready Desmos strategy
- A visual reference graph
- Common pitfalls and how to avoid them
- Practice questions in both Multiple-Choice and Free-Response formats
- A side-by-side comparison with the TI-84, where applicable

Why It's Essential

On the digital exam, graphing is more than supportive—it's strategic. The ability to graph functions, analyze features, and adjust windows confidently can make the difference between a 3 and a 5.
Mastery of Desmos isn't required—but it's your competitive advantage.

Attribution

Mission Critical

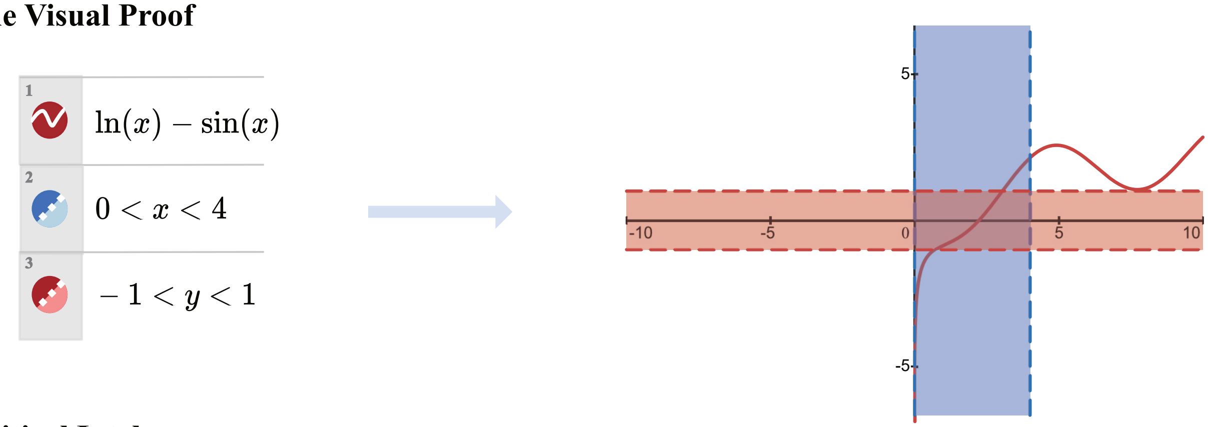
This is your first line of defense in a calculator-active problem. Being able to graph a function correctly gives you these critical advantages:

- **Visual Analysis:** It allows you to see the roots, intersections, and extrema of a function at a glance, especially when an algebraic solution is too difficult.
- **Behavior Detection:** It lets you instantly understand if a function is positive or negative over a given interval.
- **FRQ Dominance:** It is essential for understanding the behavior of rate functions.

The Desmos Execution

1. **Enter the Function:** Simply type the equation naturally into the expression list on the left (e.g.,$y = x^3 - 2x + 1$). Desmos will graph it instantly.
2. **Set the Domain (Optional but Recommended):** To focus on a specific interval $[a, b]$, type $\{a \le x \le b\}$ immediately after your function.
3. **Adjust the View:** Use your mouse or trackpad to pinch, zoom, and drag the viewing window until you see the key features of the graph clearly.

The Visual Proof



Critical Intel

- **Trap:** Analyzing the wrong part of the graph. If a question specifies an interval, always restrict your graph's domain to prevent confusion from behaviors outside that interval.
- **Strategy:** Desmos's "clickable points of interest" (gray dots) are your best friends. Use them to quickly identify roots, intersections, and extrema without needing a separate menu.
- **Strategy:** Remember that the AP Exam uses a specific version of Desmos. Practice with the official "Desmos for AP" tool to ensure you are only using exam-approved features.
- **Critical Reminder:** Always restrict your domain clearly to avoid analyzing irrelevant portions of the graph.

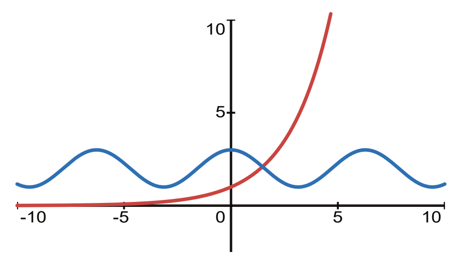
Skill #1: MCQ Practice

1. How many distinct solutions does the equation $e^{\frac{x}{2}} = \cos(x) + 2$ have?

(A) None (B) One (C) Two (D) Three

Solution & Strategy

• **Goal:** To solve an equation of the form $f(x) = g(x)$, we will graph both functions and find their points of intersection.



• **Desmos Execution:**

1. In the first line, enter $y = e^{\frac{x}{2}}$
2. In the second line, enter $y = \cos(x) + 2$
3. Analyze the graph. The two curves intersect at one distinct point.

• **Conclusion:** There are one solution. The correct answer is (B).

2. A particle's velocity is given by $v(t) = t^3 - 8t^2 + 15t$. For the time interval $0 \leq t \leq 5$, on how many open intervals is the particle moving to the left?

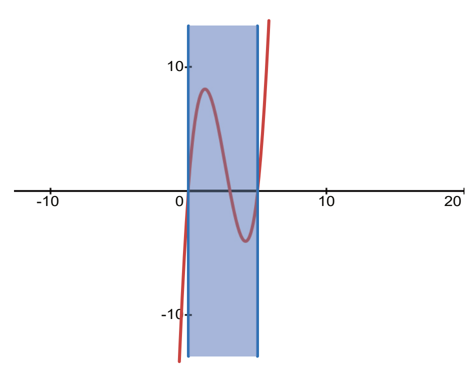
(A) None (B) One (C) Two (D) Three

Solution & Strategy

• **Goal:** The particle moves to the left when its velocity $v(t)$ is negative. We need to find the intervals where the graph of $v(t)$ is below the t-axis.

• **Desmos Execution:**

1. Enter the function $y = x^3 - 8x^2 + 15x$.
2. Restrict the domain by typing $\{ 0 \leq x \leq 5 \}$ after the function.
3. Analyze the graph. The graph is above the axis from (0, 3) and below the axis from (3, 5).



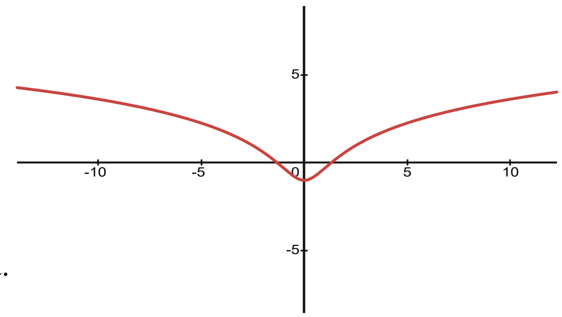
• **Conclusion:** There is one interval, (3, 5), where the velocity is negative. The correct answer is (B).

3. The derivative of a function g is given by $g'(x) = \ln\left(x^2+1\right) - 1$. How many local extrema does the original function $g(x)$ have?

(A) None (B) One (C) Two (D) Three

Solution & Strategy

- **Goal:** A function $g(x)$ has a local extremum where its derivative, $g'(x)$, changes sign (crosses the x-axis). We need to find how many times the graph of $g'(x)$ crosses the x axis.
- **Desmos Execution:**
  1. Enter the function $y = \ln\left(x^2+1\right) - 1$.
  2. Analyze the graph. The graph crosses the x-axis at two distinct points.



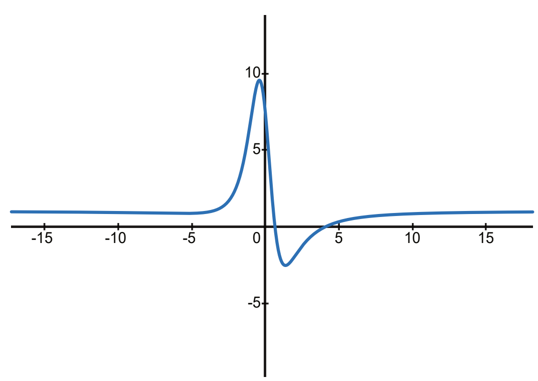
- **Conclusion:** Since the derivative $g'(x)$ changes sign twice, the original function $g(x)$ has two local extrema. The correct answer is (C).

4. For the function $f(x) = \dfrac{x^3 - 10x^2 + 5x}{x^2+2}$, the graph has how many points of inflection?

(A) None (B) One (C) Two (D) Three

Solution & Strategy

- **Goal:** A function $f(x)$ has a point of inflection where its second derivative, $f''(x)$, changes sign. We need to find the number of local extrema on the graph of $f'(x)$, as these correspond to the zeros of $f''(x)$.
- **Desmos Execution:**
  1. Define the original function: $f(x) = \dfrac{x^3 - 10x^2 + 5x}{x^2+2}$
  2. On a new line, graph the derivative: $y = \dfrac{d}{dx}f(x)$
  3. Analyze the graph of the derivative. It has three local extrema (peaks and valleys).



- **Conclusion:** Since the graph of $f'(x)$ has three local extrema, $f''(x)$ must have three zeros where it changes sign. Therefore, $f(x)$ has three points of inflection. The correct answer is (C).

Category C: Advanced and BC Skills

Skill #12 Graphing Polar Equations (BC)

BC

Mission Critical

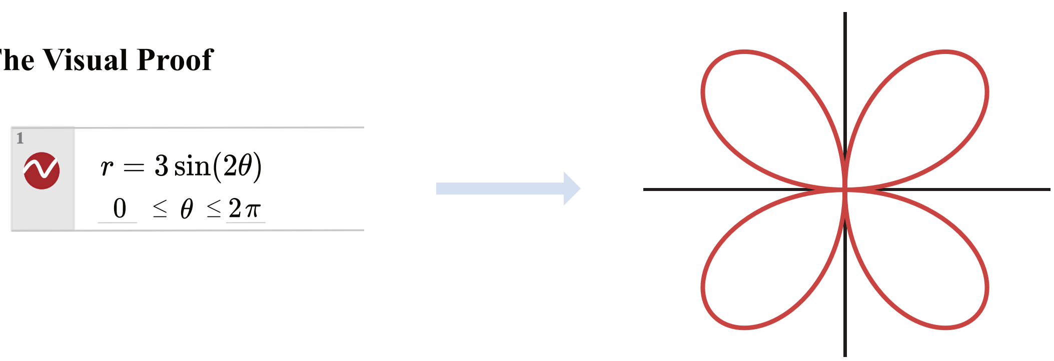
Polar equations create unique graphs like cardioids, limaçons, and roses that are often difficult to visualize without a calculator. Graphing them is essential for BC students to:

- Find the points of intersection between two polar curves.
- Determine the correct bounds of integration for finding the area of a polar region.
- Understand the path traced by a polar function over a given interval of θ.
- Visually confirm the location of vertical or horizontal tangents.

The Desmos Execution

1. **Enter the Polar Equation:** Type the equation directly into the expression list using r and θ. To type θ, you can write theta.
    - Example: $r = 1 + 2\cos(\theta)$.
2. **Set the $\boldsymbol{\theta}$ Interval:** Desmos will automatically create an input to set the interval for θ. For a complete graph of most curves, the standard interval is from 0 to 2π.
3. **Analyze the Graph:** The graph will be drawn instantly. You can click on the graph to trace along it and see the r and θ values for any point.

The Visual Proof



Critical Intel

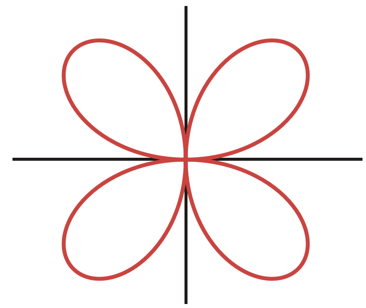
- **Trap:** Using an incorrect θmax. For many rose curves with an even coefficient (e.g., $r = \sin 2\theta$ or $\cos 4\theta$), you must set θmax to **2π** to see the full graph. Setting it to π will only draw half of the petals.
- **Strategy:** When finding the area of a region bounded by two curves, graph both r_1 and r_2 simultaneously. Click on their intersection points to find the θ values you will need for your bounds of integration.
- **Strategy:** Use the TRACE feature by clicking and dragging along the curve. This helps you understand how the curve is drawn over the θ interval, which is crucial for setting up area integrals correctly.
- **Critical Reminder:** Always confirm the initial value clearly, as small errors propagate significantly.
- **Common Mistake:** Mixing derivative (graph slope) with integral (area under the graph) interpretations.

Skill #12: MCQ Practice

1. Which of the following describes the graph of the polar equation $r = 3\sin(2\theta)$?

(A) A circle (B) A cardioid (C) A 4-petal rose (D) A limacon with an inner loop

Solution & Strategy



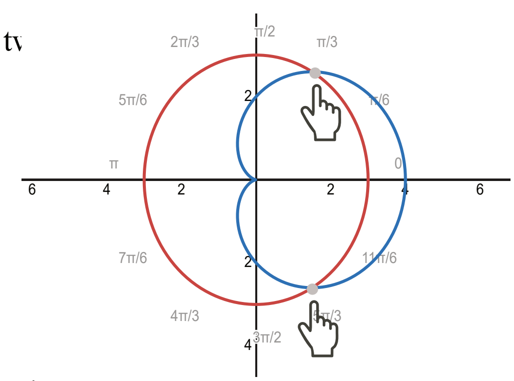
- **Goal:** To identify the shape of a polar graph.
- **Desmos Execution:**
  1. In the expression list, type $r = 3\sin(2\theta)$
  2. Ensure the θ interval is set from 0 to 2π to capture the full graph.
  3. Observe the resulting shape.
- **Conclusion:** Desmos will draw a rose curve with four distinct petals.

  The correct answer is (C).

2. At how many distinct points do the graphs of the polar curves $r = 3$ and $r = 2 + 2\cos(\theta)$ intersect?

(A) One (B) Two (C) Three (D) Four

Solution & Strategy



- **Goal:** To find the number of intersection points between tv
- **Desmos Execution:**
  1. Enter the first function: $r = 3$.
  2. On a new line, enter the second function: $r = 2 + 2\cos(\theta)$.
  3. Visually count the number of points where the circle and the cardioid cross.
- **Conclusion:** The graphs clearly intersect at two distinct points.

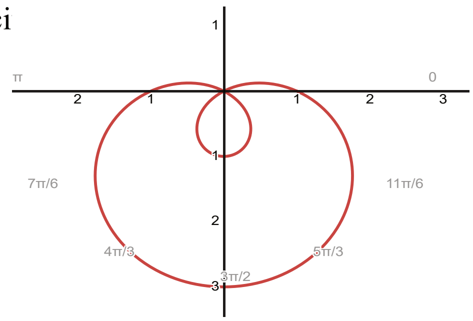
  The correct answer is (B).

3. The graph of the polar equation $r = 1 - 2\sin(\theta)$ is a limacon with an inner loop. The entire inner loop is traced on which of the following θ intervals?

(A) $[0, \pi]$ (B) $[\pi/6, 5\pi/6]$ (C) $[0, 2\pi]$ (D) $[-\pi/6, 7\pi/6]$

Solution & Strategy

- **Goal:** To find the θ interval that traces a speci
- **Desmos Execution:**



1. Enter the function: $r = 1 - 2\sin(\theta)$.
2. The inner loop is traced when r passes through zero. Find the zeros by clicking the gray dots on the origin or by solving $1 - 2\sin(\theta) = 0$. This gives $\sin(\theta) = 1/2$, so $\theta = \pi/6$ and $\theta = 5\pi/6$.
3. To confirm, set the θ slider interval from $\pi/6$ to $5\pi/6$. Observe that only the inner loop is drawn.

- **Conclusion:** The inner loop is traced on the interval $[\pi/6, 5\pi/6]$.

The correct answer is (B).

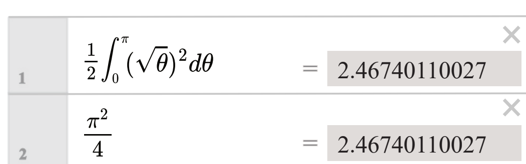
4. The area of the region enclosed by the graph of the polar curve $r = \sqrt{\theta}$ for $0 \leq \theta \leq \pi$ is:

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{2}$ (C) $\frac{\pi}{2}$ (D) π

Solution & Strategy

- **Goal:** To find the area of a region enclosed by a polar curve, we use the formula $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$
- **Desmos Execution:**

1. We need to calculate $\frac{1}{2}\int_0^{\pi}(\sqrt{\theta})^2 d\theta$
2. In Desmos, type $\frac{1}{2}\int_0^{\pi}(\sqrt{\theta})^2 d\theta$



- **Conclusion:** Desmos calculates $\frac{1}{2}\left[\frac{\theta^2}{2}\right]_0^{\pi}$,

which is $\frac{1}{2}\left(\frac{\pi^2}{2}\right) = \frac{\pi^2}{4}$.

The correct answer is (A).

FRQ in Action

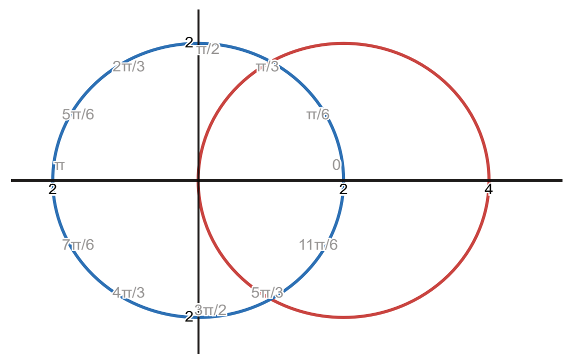
Question Snippet: Let R be the region in the first quadrant that is inside the graph of the polar curve $r = 4\cos(\theta)$ and outside the graph of the polar curve $r = 2$.

(Part a) Sketch the two polar curves and shade the region R.

(Part b) Set up, but do not evaluate, an integral expression that gives the area of R.

Solution:

1. **Identify the Goal:** We need to find the area between two polar curves, which requires finding their intersection points to determine the bounds.
2. **Graph to Visualize and Find Bounds:**
   - In Desmos, enter $r = 4\cos(\theta)$ and $r = 2$.
   - They intersect in the first quadrant. Click the gray dot at the intersection point. Desmos shows the coordinates, but more importantly, you can solve for θ.
   - Set $4\cos(\theta) = 2$, $\cos(\theta) = 1/2$. In the first quadrant, this occurs at $\theta = \pi/3$.
   - The region starts at $\theta = 0$ and ends at this intersection.



3. **Set up the Integral:**
   - The area formula is $A = \frac{1}{2}\int_{\alpha}^{\beta}\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)d\theta$.
   - In this region, the outer curve is $r = 4\cos(\theta)$ and the inner curve is $r = 2$.

- **Conclusion:** The area of R is given by the integral: $A = \frac{1}{2}\int_{0}^{\pi/3}\left((4\cos\theta)^2 - (2)^2\right)d\theta$

Graphing the curves is the essential first step to correctly identify the bounds and which function is the outer radius.

Legacy Guide: TI-84 Keystrokes

1. Set Mode: Press [MODE]. Use the arrow keys to highlight **POLAR** and press [ENTER].

2. Enter Functions: Press [Y=]. The editor will show r_1=, r_2=, etc. Enter your polar function(s). The variable key will now produce a θ.

3. Set Window: Press [WINDOW]. Critically, you must set the θ_{min}, θ_{max}, and θ_{step} values. For a complete graph, $\theta_{min} = 0$ and $\theta_{max} = 2\pi$ (or 6.28...) is standard. A good θ_{step} is $\pi/24$

The Pinnacle of Human Achievement

At its most advanced level, calculus is woven into the technologies that define modern life.

Space Exploration: Rockets and satellites rely on integrals to model fuel usage, velocity, and trajectory.

Medical Imaging: MRI and CT scans reconstruct images using integral transforms.

The Internet: Every signal, compression, and data stream is analyzed with calculus-based Fourier techniques.

What begins as solving derivatives and integrals in a classroom expands into tools that shape medicine, technology, and communication. By mastering these final challenges, you stand on the shoulders of centuries of discovery—ready to create your own ripple effect in the world.

FRQ Strategy Rooms

Train Like a 5-Scorer, Think Like a Grader

Why is This Section?

This is not just a set of practice problems — it's a system built to help you master the 10 most common FRQ missions you'll face on the AP Calculus exam.

Each "Room" presents a unique 2-page challenge, each with a different Free-Response scenario based on real College Board trends.

Why It Matters

The FRQ section makes up nearly 50% of your AP score.

Students often lose points not because of weak math skills, but due to:

- Misidentifying the question type
- Missing critical units or justification steps
- Failing to structure answers effectively

This section trains all three.

What You'll Gain

- Instant recognition of FRQ patterns
- A scoring-focused mindset
- Structured, confident answers under pressure

How to Approach Each Room

1. Before solving, review the Briefing and Kit.
2. As you solve, use the Battle Plan to guide structure and language.
3. After solving, study the full solution and scoring annotations.
4. Repeat a few days later for retention and pattern fluency.

A Strategic Reminder

FRQs don't just test what you know — they test how well you communicate and justify.
This section is your training ground for both.

1. How to Recognize This Mission (Signals)

Given: The problem provides the graph of $f'(x)$, not $f(x)$. This is the most important signal.

Graph Style: The graph is often composed of simple geometric shapes like line segments, semicircles, or quarter circles, allowing for exact area calculations.

Typical Questions: You will be asked about the properties of the original function f, such as:

- "When is f increasing?"
- "Where is f concave down?"
- "Find the value of $f(b)$ given $f(a)$."

2. Required Tools (Concepts Needed)

Derivative–Function Connection:

- $f' > 0 \Leftrightarrow f$ is increasing.
- $f' < 0 \Leftrightarrow f$ is decreasing.
- f' changes from negative to positive $\Leftrightarrow f$ has a local minimum.
- f' changes from positive to negative $\Leftrightarrow f$ has a local maximum.

Second Derivative Connection:

- f' is increasing $\Leftrightarrow f'' > 0 \Leftrightarrow f$ is concave up.
- f' is decreasing $\Leftrightarrow f'' < 0 \Leftrightarrow f$ is concave down.
- The slope of f' changes sign $\Leftrightarrow f$ has a point of inflection.

Integral–Function Connection (FTC):

- The net change in f is the area under the f' curve: $f(b) = f(a) + \int_a^b f'(x)dx$

3. Scoring Steps (The Game Plan)

Step 1 (Increasing/Decreasing):

Identify the intervals where the f' graph is above the x-axis ($f' > 0$) or below the x-axis ($f' < 0$).

Step 2 (Concavity):

Identify the intervals where the f' graph itself is increasing (slope of f' is positive) or decreasing (slope of f' is negative).

Step 3 (Finding Values):

Use the given initial condition (e.g., $f(a) = k$) and find the areas of the geometric shapes on the f' graph to calculate values of $f(x)$ at other points using the FTC.

Step 4 (Extrema):

To find absolute extrema, use the Candidates Test. Identify the endpoints of the interval and all critical points (where $f' = 0$ or changes sign).

Calculate the value of f at each of these candidate points using the area method from Step 3. The largest value is the absolute max, the smallest is the absolute min.

4. Common Traps

Confusing f with f':

When asked for where f is increasing, look at where f' is positive, not where f' is increasing.

Forgetting the Initial Value:

When calculating $f(b)$, don't forget to add the initial value $f(a)$. The integral $\int f'(x)\,dx$ only gives you the *change* in f, not the final value.

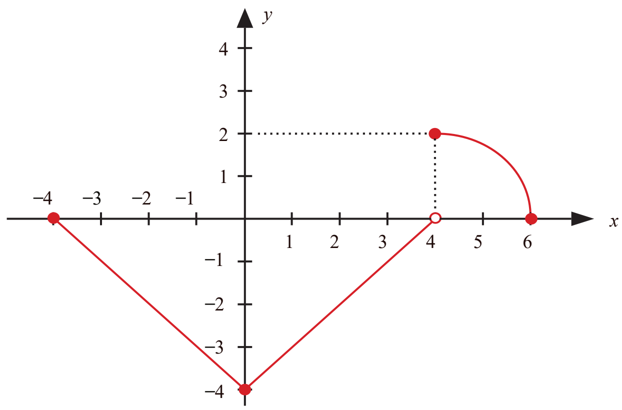
Confusing Extrema with Inflection Points:

An extremum of f occurs where f' crosses the x-axis (changes sign).

An inflection point of f occurs where f' has a *peak or valley* (changes direction/slope).

(Mission 1: Practice FRQ 1)

Instructions: This is a non-calculator question. Show all your work and justify your answers.



Question: The graph of f', the derivative of a twice-differentiable function f, is shown above for the interval $[-4, 6]$. The graph consists of two line segments and a quarter circle centered at $(4, 0)$. It is known that $f(0) = 5$.

(a) Find the value of $f(4)$ and $f(6)$.

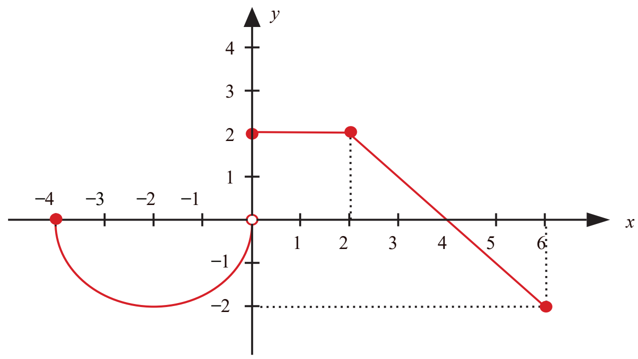
(b) On what open intervals is the graph of f both decreasing and concave up? Give a reason for your answer.

(c) Find the x-coordinate(s) of the local minimum(s) of the function f. Justify your answer.

(d) Find the absolute maximum value of f on the closed interval $[-4, 6]$. Justify your answer.

(Mission 1: Practice FRQ 2)

Instructions: This is a non-calculator question. Show all your work and justify your answers.



Question: The graph of f', the derivative of a twice-differentiable function f, is shown above for the interval $[-4, 6]$. The graph consists of two line segments and a semicircle. It is known that $f(-4) = 3$.

(a) Find the value of $f(0)$ and $f(2)$.

(b) On what open intervals is the graph of f both increasing and concave down? Give a reason for your answer.

(c) Find the x-coordinate(s) of the point(s) of inflection of the function f. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-4, 6]$. Justify your answer.

1. How to Recognize This Mission (Signals)

Given: The problem will describe a population whose growth slows down as it approaches a maximum capacity. The differential equation will be given in one of two specific logistic forms:

1. $\frac{dP}{dt} = kP(L-P)$ or 2. $\frac{dP}{dt} = rP\left(1-\frac{P}{L}\right)$

Keywords: You will see phrases like **"carrying capacity"**, "population growth," and an initial condition.

Typical Questions: You'll be asked to find the **carrying capacity** L, find the population value when the growth rate is at its maximum, and solve the differential equation for a particular solution.

2. Required Tools (Concepts Needed)

Carrying Capacity (L):

This is the maximum sustainable population, or the value that $P(t)$ approaches as t → ∞. You can identify L directly from the given differential equation.

Maximum Growth Rate:

This is a crucial fact you must memorize. A logistic population grows **fastest** when the population is at **half its carrying capacity**: $P = \frac{L}{2}$

Solving the Differential Equation:

This is an application of the **"Separation of Variables"** technique,but it always requires **partial fraction decomposition** to integrate the population side. **The separation leads to an integral of the form**:

$$\int \frac{1}{P(L-P)}dP = \int kdt$$

Second Derivative Analysis:

The solution curve for logistic growth has a point of **inflection** at $P = \frac{L}{2}$ which is where the graph changes from concave up to concave down.

3. Scoring Steps (The Game Plan)

Step 1 (Identify L and Max Growth Point):

As soon as you see the logistic equation, identify the **carrying capacity *L***.
If the question asks for when the population grows fastest, immediately state: $P = \frac{L}{2}$
These are often easy, quick points.

Step 2 (Calculate Rates):

If asked for the rate $\frac{dP}{dt}$ at a specific population P, simply plug the value of P into the given differential equation.

Step 3 (Find the Particular Solution):

This is the most algebra-intensive part:

1. **Separate** the variables.
2. Use **partial fractions** to decompose the term with P.
3. **Integrate** both sides and add $+C$.
4. Use the **initial condition** to solve for the constant C.
5. **Solve for *P*** explicitly.

This will usually involve using properties of logarithms and exponentials.

4. Common Trap

Misidentifying the Carrying Capacity L:

Be careful with the form of the equation. If it's given as $\frac{dP}{dt} = 0.05P(200 - P)$, then $L = 200$. If it's given as $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{200}\right)$, then L is also 200.

Algebraic Errors:

The partial fraction decomposition and the final step of solving for P are prone to algebraic mistakes. Work carefully and step-by-step.

Confusing "When" and "What Value":

The question "At what population value is the growth rate fastest?" asks for $P = \frac{L}{2}$. The question "At what time is the growth rate fastest?" asks for the t value, which you can only find after solving for $P(t)$.

(Mission 10: Practice FRQ 1)
Instructions: This is a non-calculator question. Show all your work and justify your answers.

Question: A population of bears on a remote island is modeled by the function $P(t)$, where P is the number of bears at time t in years. The population satisfies the logistic differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P\left(1 - \frac{P}{400}\right)$$

At time $t = 0$, the population is $P(0) = 50$.

(a) What is the carrying capacity for the bear population on the island? At what population size is the bear population growing the fastest?

(b) What is the rate of change of the population, $\frac{dP}{dt}$, at time $t = 0$

(c) Find the particular solution $P(t)$ to the differential equation with the initial condition $P(0) = 50$.

(d) Use your solution from part (c) to find the time t when the population reaches 100 bears.

(Mission 10: Practice FRQ 2)
Instructions: This is a non-calculator question. Show all your work and justify your answers.

Question: The number of students at a new school, $S(t)$, is modeled by the logistic function

$$S(t) = \frac{1200}{1 + 3e^{-t}}$$

where t is the time in years since the school opened.

(a) What is the number of students when the school first opens?

(b) Find the carrying capacity of the school. At what population size is the school's student population growing the fastest?

(c) Find $\frac{dS}{dt}$ at the time when the population is growing fastest.

(d) Find the particular differential equation that models this situation.